THE MIGRATION OF MATHEMATICIANS

ARNOLD DRESDEN, Swarthmore College

In 1785, Joseph Priestley was elected to membership in the American Philosophical Society. In June 1794, at the age of 61, he came to America and settled in Northumberland, Pennsylvania. At its meeting of June 20, 1794, the American Philosophical Society appointed a committee to prepare a congratulatory address to Dr. Priestley. At the same meeting the committee reported a draft, which was adopted "and the officers of the Society with as many of the other members as can conveniently attend are directed to meet at the Hall tomorrow afternoon at 1 o'clock in order to present the same." In the minutes of the meeting of July 18, 1794 it was "reported that a number of the officers and members . . . waited on Dr. Priestley 'and presented the address.'" It is as follows:

To Joseph Priestley, LL.D., &c.

The American Philosophical Society, held at Philadelphia, for Promoting Useful Knowledge, offer you their sincere congratulations on your safe arrival in this country. Associated for the purposes of extending and disseminating those improvements in the sciences and the arts, which most conduce to the substantial happiness of man, the Society felicitate themselves and their country, that your talents and virtues have been transferred to this Republic. Considering you as an illustrious member of this institution, your colleagues anticipate your aid, in zealously promoting the objects which unite them; as a virtuous man possessing eminent and useful acquirements, they contemplate with pleasure, the accession of such worth to the American Commonwealth; and looking forward to your future character of a citizen of this your adopted country, they rejoice in greeting, as such, an enlightened Republican.

In this free and happy country, those unalienable rights, which the Author of Nature committed to man as a sacred deposite, have been secured. Here, we have been enabled, under the favour of Divine Providence, to establish a government of laws and not of men; a government, which secures to its citizens equal rights and equal liberty; and which offers an asylum to the good, to the persecuted, and to the oppressed of other climes.

May you long enjoy every blessing which an elevated and highly cultivated mind, a pure conscience, and a free country are capable of bestowing.

By order of the Society,
DAVID RITTENHOUSE *Pres.*

PHILADA, June 20th 1794

It is the spirit of this congratulatory address which I should like to recapture in this brief report on the mathematicians from abroad who have come to America in consequence of the events of the last ten years. It is meant to be congratulatory both to the newcomers and to us who were here to receive them, having come at birth or later. American mathematicians would like to wait on the good who have sought asylum from persecution and oppression, whose worth has acceded to the American Commonwealth. And this welcome is to be extended not only to those of high eminence in the profession, but to all who intend to make their contribution to American life as professional mathematicians. It is but natural that great honor is accorded to those who have already made contributions of great value and whose association with American mathematics gives more than promise for the future. But it is a truism that

rank and file men are needed as well as captains, colonels and generals. American mathematicians are "associated for the purposes of extending and disseminating" the beauty and value of their science and they "felicitate themselves and their country" that so many "talents and virtues" have become united with them.

Thus no attempt is made to evaluate the significance of the individuals whose names appear; in each group the names are placed in alphabetical order. While it is a regrettable fact that in 1942 it is still necessary to open an asylum to the persecuted and oppressed, we have good reason to congratulate ourselves that the shortage of teachers of mathematics, needed for the instruction of members of the armed forces, may be met, at least in part, by using the talents of those who have come, whatever be the level of eminence to which these talents entitle them.

No one should minimize the difficulties caused by the arrival during a short period of time of a large number of Priestleys. It requires all the resources of administrative statesmanship which we can command to make practically effective the welcome which we want to give them. This problem constitutes a challenge to the strength of our organization, and it is a source of satisfaction that to such a large extent we have succeeded in solving it.*

The record which follows indicates how far we have succeeded. It will be seen that several of the newcomers have not yet been placed; and that many more have not found situations in which they can contribute their gifts and accomplishments most effectively. Perhaps this report will help to bring about more satisfactory placements. There is no doubt that the increase in our resources is an event of the first magnitude in the history of American mathematics. If properly utilized it should give an impetus to the development of American mathematics, whose effects will carry us forward to great heights.†

As with most historical events of importance, there were forebodings of what was to come, for some years before 1933. There are many who can be looked upon as forerunners of the great migration which began in 1933. The names of Bohnenblust, von Kármán, Landé, von Neumann, Radó, Schoen-

* Through the work of the Emergency Committee in Aid of Displaced Foreign Scholars, of the Carl Schurz Memorial Foundation, of the Rockefeller Foundation, of the New School for Social Research, and of the officers of the American Mathematical Society, a great many mathematicians, unfortunately not all, have been placed in permanent positions. The author is indebted to these organizations, also to Dean Richardson of Brown University, to Professor H. Weyl of the Institute for Advanced Study in Princeton, and to many other individuals, for helping him in obtaining the information on which this account is based. In many cases the information is incomplete; it is hoped that it is substantially correct. It is quite likely, particularly for the border fields, that persons have been mentioned who do not consider themselves mathematicians. Such errors of commission are to be regretted, but less so than inadvertent omissions. The author will be grateful for the receipt of corrections and of supplementary data.

† Two articles which have an interesting bearing on the subject of this report have appeared recently in the American Scholar, Vol. 11, No. 3 (Summer, 1942), viz.: B. W. Huebsch, Cross-fertilizations in letters (pp. 304–314), and Herbert Solow, Refugee Scholars in the United States (pp. 374–378).

berg, Seidel, Shohat, Struik, Tamarkin, Uspensky, Wigner, Wintner, Zariski, have long been familiar to American mathematicians. They are men who have become completely absorbed in the scientific life of the country and who have enriched it in a significant way.

1933

The same can be said about those who arrived in 1933, the first year of the period with which we are concerned. They are:

Eugen Altschul after receiving the Ph.D. at Freiburg in 1912, was engaged in business, became lecturer in statistics and economic theory at the University of Frankfurt a/M, and spent a half year at the London School of Economics before coming to this country. He has been a lecturer at the University of Minnesota since his arrival, has undertaken a number of special studies, and is now on leave in order to carry on research for the War Production Board.

Felix Bernstein after many years of work at Göttingen, spent the years from 1933–36 at Columbia University and has since then been professor of Biometry at New York University.

S. Bochner came from Munich to Princeton after a short time spent in England.

Richard Brauer has been at the University of Toronto since 1935 (visiting professor at the University of Wisconsin for half of the year 1940–41, holder of a Guggenheim Fellowship 1941–42), having spent a year at the University of Kentucky and a year at the Institute for Advanced Study,* after coming from Königsberg.

Albert Einstein's career, both here in and Europe, and his contribution to American life are too well known to require mention here.

Fritz Herzog completed his studies for the Ph.D. at Columbia University in 1935 and has been since 1939 an instructor at Cornell University.

Hans Lewy came to this country from Göttingen, spent the years 1933–35 at Brown University and has since then been at the University of California.

Walther Mayer has been at the I.A.S. since his arrival in this country.

Emmy Noether came to Bryn Mawr College from Göttingen and remained there until her earthly career was cut short in 1935.†

Otto Szász came from Frankfurt a/M, spent the years until 1936 at the Massachusetts Institute of Technology, at Brown University and on a lecture tour. Since 1936 he has been at the University of Cincinnati.

Hermann Weyl came from Göttingen to the I.A.S., after having spent the first weeks of his permanent residence in this country as the Cooper Foundation lecturer at Swarthmore College.

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* The "Institute" will hereafter be referred to as I.A.S.

† Bryn Mawr College has published the memorial address delivered by Hermann Weyl on April 26, 1935.

Thus we see that the group which came in 1933 has a perfect record for thorough assimilation to American scientific life.

1934

An equally good record can be reported for the party which came in 1934.

Hans Bethe, mathematical physicist, left Tübingen in 1933, spent the next year in England and has been at Cornell Uiversity since his arrival in this country.

Felix Bloch, mathematical physicist in Leipzig until 1932 has been at Stanford University since 1934.

Richard Courant left the directorship of the Mathematical Institute at Göttingen in 1933, spent the year 1933–34 at Cambridge University and has been at New York University from 1934 on.

George Gamow, mathematical physicist at Leningrad until 1933 has been professor at the George Washington University since 1934.

Hans Rademacher, after earlier appointments at Berlin and Hamburg, had been professor at Breslau since 1925. He came to this country as visiting professor at the University of Pennyslvania. His position at that university was later made permanent.

Gabor Szegö came from Königsberg to Washington University in St. Louis, and went from there in 1938 to the chairmanship of the department of mathematics at Stanford University.

Stefan Warschawski had spent some years at Göttingen and a year at Utrecht before coming to the United States. A year at Columbia was followed by two years at Cornell, and two years at Rochester, before he became assistant professor at Washington University, St. Louis, in 1939.

Max Zorn, student of Artin's in Hamburg, held assistantships there and at Halle. A fellowship at Yale, until 1936, was followed by an appointment at the University of California at Los Angeles.

1935

The number of arrivals varied but slightly during the following three years. More and more the I.A.S. became a reception center for refugee mathematicians. The presence on its staff of Einstein and Weyl acted as a magnet for German physicists and mathematicians. Added to this, the great personal interest of Oswald Veblen and other members of the staff, and the active participation in measures necessitated by the events in Europe of the director of the Institute, Abraham Flexner, and his successor in 1939, Frank Aydelotte, made the Institute a natural way station for the victims of fascist and nazist persecution and for others whose emigration to this country was more nearly voluntary. In 1935 came the following persons:

Reinhold Baer was in Halle until 1933, in England for two years, at the I.A.S. during the years 1935–37. This was followed by a year at the University

of North Carolina and finally by a permanent appointment at the University of Illinois in 1938.

Herbert Busemann remained at the University of Göttingen for some years after receiving the doctorate, spent the years from 1936–39 at the I.A.S., the next year partly at Swarthmore College and partly at Johns Hopkins. Since 1940 he has been at the Illinois Institute of Technology.

Max Herzberger, theoretical physicist, who came from Jena, where he had been associated with the Zeiss Co., has become a member of the research staff of the Eastman Kodak Co.

Fritz John received the Ph.D. from Göttingen in 1933 and spent the years 1934–35 at Cambridge University. Upon arrival in this country, he became visiting professor at the University of Kentucky. He has now a permanent appointment at that institution.

Bela A. Lengyel came from the Polytechnic Institute in Budapest. He has held a fellowship at Harvard, was statistician in Worcester, Mass., during 1938 and he has been, since 1939, an instructor at the Rensselaer Polytechnic Institute.

Lothar Nordheim, theoretical physicist, in Paris after leaving Göttingen, was at Purdue University from 1935 to 1937. Since then he has been at Duke University.

O. F. G. Schilling completed his studies at Marburg in 1934, spent two years at the I.A.S. (1935–37) and two years at Johns Hopkins (1937–39). At present he holds a permanent appointment at the University of Chicago.

Leo Szilard, physicist from the Kaiser Wilhelm Institute, came first to New York University and is now at Columbia University.

E. Teller, theoretical physicist formerly at Göttingen, spent a year at London University and is now professor at George Washington University.

1936

The year 1936 was one of comparative quiet; it brought to this country a smaller number of mathematicians than any other year during the nine-year period except 1934.

P. G. Bergmann, theoretical physicist, holds the degree of Ph.D. from Prague (1936) and was an assistant at the I.A.S. for four years. He is now a member of the faculty of Black Mountain College.

W. Z. Birnbaum was on the staff of the Institute for Mathematical and Statistical Research (500 Fifth Avenue, New York), spent a year at New York University and has had an appointment at the University of Washington since 1938.

Olaf Helmer studied in Berlin until 1934, took the degree of Ph.D. in London in 1936 and then spent a year visiting a number of colleges and universities in this country and lecturing. An assistantship at the University of Chicago, 1937–38, was followed by an appointment at the University of Illinois.

W. Hurewicz holds a doctor's degree from Vienna and lectured for many years at the University of Amsterdam. He spent the years 1936–39 at the I.A.S. and has been at the University of North Carolina since 1939.

H. A. Jordan is associate professor of mathematics in the Graduate School of Georgetown University.

Eric Reissner continued at the Massachusetts Institute of Technology, from 1936 on, the studies which were begun in Berlin. In addition to the degree of Eng.D. from Berlin (1936), he holds the degree of Ph.D. from M. I. T. (1938). After having been an assistant and instructor, he is now an assistant professor of mathematics at that institution.

Hertha Sponer, theoretical physicist at Göttingen, spent two years at Oslo and has been at Duke University since 1936.

S. Ulam obtained the degree of D.Sc. at Lwów in 1933 and carried on further studies in Zürich and in Cambridge. After spending part of a year at the I.A.S., he was a member of the Society of Fellows at Harvard University (1936–39) and a lecturer at Harvard during 1939–40. Since 1941 he has been a member of the faculty at the University of Wisconsin.

1937

During 1937, a somewhat larger group arrived in this country including some very well known mathematicians; this group consisted of the following persons:

E. Artin came from Hamburg, was at Notre Dame University until 1938 and went from there to a permanent position at the University of Indiana.

Valentin Bargmann, theoretical physicist, holds the doctorate from Zürich (1936) and has been at the I.A.S. since his arrival in this country. From 1940 on he has been assistant to Einstein.

Kurt Friedrichs came from a professorship at Braunschweig, having previously lectured at Aachen and at Göttingen. He became visiting professor at New York University, where he has been on a permanent appointment since 1940.

Hans Hertz was a student at Hamburg from 1934 to 1937. He continued his studies in astronomy at Yale, where he received the Ph.D. degree in 1941. He has been an assistant at the Yale Observatory from 1940 on and he held a Sterling Research fellowship during the year 1941–42.

Leopold Infeld, theoretical physicist, had been a professor at Lwów until 1936. He spent a year at the I.A.S. and obtained a permanent appointment at the University of Toronto in 1938.

Michael Lotkin supplemented his work at Kiel, where he received a doctor's degree, by courses at New York University and he is now teaching at the Tilden High School in Brooklyn.

Karl Menger came to Notre Dame University after having lectured at the University of Vienna for over 10 years.

Erich Rothe holds the Ph.D. from Berlin (1927), lectured at Breslau until 1935, has been at Wm. Penn College (Oskaloosa, Ia.) since 1937.

Feodor Theilheimer holds the Ph.D. from Berlin (1936) and taught in secondary schools there for a year. He has been engaged in private tutoring in St. Louis, 1937–41, and has taken part during the past year in the advanced course in mechanics at Brown University.

1938

The next year, 1938, brought a very considerable rise of the tide.

F. L. Alt was engaged in actuarial work in Vienna, where he had received the doctor's degree. He is at present statistician with the Institute of Applied Econometrics (500 Fifth Avenue, New York).

Alfred Bloch had been engaged in secondary school work in Germany, after having studied at Strassburg and Munich and having passed the "Staatsexamen" in mathematics and physics in 1922. He has been teaching, chiefly German language, at the University of Utah.

Claude Chevalley holds a doctorate from Paris and has lectured at Strassburg and at Rennes. He was at the I.A.S. during the year 1938–39 and has been since then at Princeton.

Paul Erdös has a Ph.D. from Budapest (1934) and the D.Sc. degree from Manchester (1938). He spent four years in England before coming to this country, was at the I.A.S., 1938–40, and has held a fellowship at the University of Pennsylvania since 1940.

Philipp Frank, theoretical physicist at Prague for many years, is now at Harvard University.

Kurt Gödel had been at the I.A.S., while still on the faculty of the University of Vienna, during the year 1933–34 and part of 1935–36. He is now a member of the I.A.S., and lectured at Notre Dame University during part of the year 1938–39.

Eduard Helly is a Ph.D. from Vienna (1907), taught in the secondary schools and lectured at the University of Vienna. He has held important positions in the actuarial field and as consultant with financial institutions. He taught at the Paterson Junior College, 1940–42 and has now a position at Monmouth Junior College (Long Branch, N.J.).

Mark Kac is a Ph.D. from Lwów (1937), where he was a research assistant, 1935–37. He was also engaged in actuarial work in Europe. He held a fellowship at Johns Hopkins during the year 1938–39 and has been an instructor at Cornell University from 1939 on.

Gerhard Kalisch completed in 1941 at the University of Chicago the studies begun in Europe. A fellowship at Chicago, 1939–41, was followed by a year at the I.A.S. and by an appointment at the University of Kansas.

Jacob Klein, a Ph.D. from Marburg (1922), lectured at the University of Prague, 1934–35, and held a special fellowship in Berlin, 1935–37. He is now teaching at St. John's College.

Eugen Luckacs holds a doctor's degree from Vienna (1930). He was engaged in actuarial work in Vienna and Trieste, and taught in the schools and in the

people's university in Vienna. According to available information, he lives at present in Baltimore.

H. B. Mann, Vienna Ph.D. (1935), has taught in secondary schools and privately in Vienna. After a number of positions and private tutoring, he has now a temporary research associateship at Columbia University.

Karl Meissner, physicist, formerly at Frankfurt a/M, spent two years at Worcester Polytechnic Institute and is now at Purdue University.

Paul Nemenyi holds the D.Sc. degree from Berlin (1922) and has lectured on various engineering subjects at technical schools in Berlin and in Copenhagen. He lectured at a number of places in this country, was engaged in hudraulic research in Iowa and has been an instructor at Colorado State College since 1941.

Jerzy Neymann came from an active career in statistical work in Polish universities to the University of California in Berkeley.

Hans Reissner, holder of degrees from Berlin and Aachen, was professor at the technical universities of Aachen (1906–12) and Berlin (1912–36). He is now research professor of Engineering at the Illinois Institute of Technology.

Helene Reshovsky received the degree of Ph.D. in Vienna in 1930 and was a teacher of mathematics and physics in a "Realgymnasium" in that city. During the years 1939–41 she taught the same subjects at Dana Hall School (Wellesley, Mass.). At present she holds a similar position at the Baldwin School (Bryn Mawr, Pa.).

M. A. Sadowsky has been at the Illinois Institute of Technology since his arrival in this country. A holder of the doctorate from Berlin-Charlottenburg (1927), he taught at this institution from 1926–31, at the University of Minnesota, 1931–33 and at Leningrad and Novocherkassk, 1934–37.

Catharine Stern took the Ph.D. degree at Breslau in 1918, her major interest being mathematics and science. She has been largely concerned with educational questions, bearing on the training of children, both in Europe and here. She is associated with the Child Study Association of America and the American Psychological Association. She is included in this report because her work may have an important influence on the teaching of mathematics in our schools.

Abraham Wald received the degree of Ph.D. in Vienna in 1930, was assistant of Karl Menger, 1930–34, member of the Institute of Business Cycle Research (Vienna) during the years 1934–38. After holding a fellowship of the Cowles Commission during the summer of 1938, he became successively research associate in mathematical statistics (1938–40), lecturer (1940–42), and assistant professor at Columbia University.

1939

The peak of the immigration of mathematicians was reached in 1939, a year after the start of increased persecutions in Europe, the year in which the war broke out. Many of those who came in 1939 had of course made preparations long in advance, so that this high tide reflects events of the preceding years

probably to a larger extent than those of 1939. The reader will recognize in the list which follows several names which were well known among American mathematicians for a long time, and which have acquired added importance for our scientific development.

Alfred Basch obtained the doctorate from the Vienna Technical University and has combined teaching at technical schools in Dresden, Prague and Vienna with engineering practice. He taught at Holy Cross College (Worcester, Mass.) during the years 1939–42. At present he has a position in the Paterson Junior College and, during the summer of 1942, in the Harvard Summer School of Engineering.

Gustav Bergmann, theoretical physicist (Ph.D., Vienna, 1928) and a doctor of jurisprudence (Vienna, 1935), was a lecturer and assistant in mathematics at the University of Vienna, and practised law in Vienna during the years 1935–38. He has been at the University of Iowa since 1939.

Stefan Bergmann has a doctor's degree from Berlin (1922), has taught at Berlin and at the Technological Institute of Tomsk. He lectured at the Massachusetts Institute of Technology, 1939–40, and is at present visiting lecturer in applied mathematics at Brown University.

Alfred T. Brauer was assistant to I. Schur at the University of Berlin, 1926–35 and took his degree there in 1928. He has been at the I.A.S. during the years 1939 to 1942, contributing a great deal to the formation of the library of the Institute's school of mathematics. He has now a temporary appointment at the University of North Carolina.

Walther Bruns was known in the German educational world as Walther Jacobsthal. He occupied important positions as director and administrator of schools in Berlin and as a lecturer on mathematical education. He is at present living in Syracuse, N. Y.

Max Chameides was a teacher in the schools and in the "Volksuniversitat" in Vienna.

Samuel Eilenberg received the Ph.D. degree at Warsaw in 1936, spent the next three years in Paris and Cambridge, and has been at the University of Michigan since 1939, obtaining appointment to the regular faculty in 1941.

Willy Feller, Ph.D. Göttingen, was at the University of Kiel, 1928–33, and at Stockholm during the years 1934 to 1939. Since his arrival in this country, he has been at Brown University.

Guido Fubini, professor at the University of Turin for over 30 years, has been at the I.A.S. since 1939.

Hilda Geiringer holds a Ph.D. from Vienna (1918) and has had wide experience in probability, statistics and various other fields of applied mathematics, at the Universities of Berlin, Brussels and Istanbul. She has been on a temporary appointment at Bryn Mawr College, 1939–42, and has filled a part-time position at Swarthmore, 1941–42. During the summer of 1942, she is on the staff of the School of Mechanics at Brown University.

Michael Golomb received the Ph.D. in Berlin in 1934 and spent the next five

years in Yugoslavia. Since coming to this country he has been at Cornell University as a research assistant and as a part-time instructor. In September 1942 he goes to Purdue University as instructor.

Ernst Hellinger, professor at Frankfurt a/M from 1914 to 1938, has been a visiting lecturer at Northwestern University since 1939.

Walter Jacoby studied at the University of Berlin and was a teacher in the high schools of that city.

Georg Jaffé, professor of theoretical physics at Leipzig, 1905–26, and at Giessen, 1926–33, is at present on the faculty of Louisiana State University.

Arthur Korn, professor at Berlin since 1913 and active in various fields of applied mathematics, is now at Stevens Institute of Technology.

Friedrich Kottler, professor of theoretical physics at Vienna, 1919–38, was a member of the staff of the Eastman Kodak Company, 1939–40.

Gustav Kürti, Ph.D. in physics from Vienna, was at the University of Rochester and spent the year 1941–42 at the Massachusetts Institute of Technology.

Gustav Land (formerly Gustav Deutschland), Berlin Ph.D. in Astronomy (1908), was active in astronomy and in meteorology at the observatories in Berlin, Königsberg and Leipzig until 1919. He was engaged in statistical work until 1938. After a short period at the Nautical Almanac Office in London, he became Research Associate at the Sproul Observatory (Swarthmore), 1939–41. Since then he has been research assistant and instructor at the Yale Observatory.

Gerhard Lewin, physicist with a Ph.D. from Prague (1930), was engaged in industry and business in Berlin, 1930–33, and in Prague, 1933–37. Since 1939, he has been connected with the R.C.A. laboratory in Harrison, N. J.

Karl Loewner received the Ph.D. in Prague (1917) and lectured in Berlin (1922–28), in Cologne (1928–30) and in Prague (1930–38). Since 1938 he has been visiting professor at the University of Louisville.

H. T. Ludloff was a privat-dozent at the University of Breslau. He spent some time at Thiel College (Greenville, Pa.) and is now in the department of physics at City College, N. Y.

Richard von Mises was for many years (1920–33) professor of applied mathematics at Berlin, later at Istanbul (1934–39). He is now visiting professor at the Graduate School of Engineering of Harvard University, lecturer at the Massachusetts Institute of Technology, and during 1941–42, visiting professor at Brown University.

Otto Neugebauer, privat-dozent (1926–32) and later professor at Göttingen (1932–34), spent the years 1934–39 at the University of Copenhagen, continuing there the editorship of the Zentralblatt für Mathematik, which he had started in 1931. Since 1939, he has been professor at Brown University and editor of Mathematical Reviews.

Gottfried Noether, nephew of Emmy Noether, carried on in this country the studies begun in Breslau and continued in Tomsk, at Ohio State Univer-

sity (1939–40) and at Illinois (1940–41). He is now in the U. S. Army.

I. Opatowski had spent the years 1932–35 at the University of Turin and was later employed by the Fiat Co. (1936–38). He was an instructor at the University of Minnesota (1939–42) and is now at the Armour Research Foundation, Illinois Institute of Technology.

Anita Riess studied at Heidelberg, Leipzig and Marburg, and received the Ph.D. from the latter university in 1921. She has been active as an educational administrator and as a teacher in secondary schools and normal schools in Leipzig and in Hamburg. She is at present a visiting lecturer at Wellesley College.

Peter Scherk received the Ph.D. degree at Göttingen in 1935. The year 1939–40 was spent at the Taft school. The next year he held a fellowship at Yale and during 1941–42 he was at the University of Indiana.

Olaf Schmidt was a student at the University of Copenhagen and a teacher in the secondary schools. He has been at Brown University since his arrival in this country as an instructor and as a research assistant to Professor Neugebauer.

Alfred Seckel, active as teacher and as educational administrator (Staatsexamen in philosophy, mathematics and physics) in Magdeburg and in Freiburg, had a position at Canisius College (Buffalo) from 1939 till 1941. He is now at the Chestnut Hill Academy (Philadelphia).

Wolfgang Sternberg was professor at Heidelberg during the years 1920–27 and at Breslau during the years 1927–33. At present he is living in New York City.

Alfred Tarski, professor at Warsaw, 1925–39, has been a visiting lecturer at Harvard University (1939–41) and at New York City College. He held a Guggenheim Fellowship during the year 1941–42 and has been called to fill a temporary vacancy at the University of California at Berkeley for 1942–43.

Otto Treitel was a secondary school teacher in Mannheim and in Heidelberg. He has been associated with Milton College.

Wolfgang Wasow was a student at Göttingen (1930–33) and at Paris (1933–34). He taught in German schools in Italy during the years 1935–38. A part of the year 1939 was spent at the Choate School (Wallingford, Conn.), the next year at Goddard Junior College (Plainfield, Vt.). During the years 1940–42 he has held a fellowship at New York University and since 1941 he has taught at the Connecticut College for Women.

Alexander Wundheiler received the degree of Ph.D. at the University of Warsaw in 1932 and lectured there from 1927–39. He held a special lectureship at the Massachusetts Institute of Technology during the second semester of 1940 and taught mathematics at Tufts College during the following year. Since February 1942 he has been an instructor in the physics department at New York City College.

Antoni Zygmund, professor at Warsaw (1922–30) and at Wilno (1930–39), has been on the faculty of Mt. Holyoke College since 1939. He has also been a lecturer at the Massachusetts Institute of Technology and a visiting professor at the University of Michigan.

1940

The next year, 1940, showed a considerable decrease, due to a large extent very probably to the increasing difficulties of communication and of transportation. The arrivals during 1940 include the following:

Felix Adler, theoretical physicist with a Ph.D. from Zürich (1938), spent the years 1938–40 at the Collège de France. He was at the I.A.S. during the year 1941–42 and has a temporary appointment at the University of Wisconsin for 1942–43.

Hugo Basch, an engineer holding diplomas from the Vienna Technical University, was engaged in important civil engineering enterprises in Austria. He has been associated with engineering firms in Philadelphia since 1940 and lives in Moorestown, N. J.

Lipman Bers received the Ph.D. from Prague in 1938 and spent the years 1938–40 in Paris. He took part in the work of the advanced mechanics courses at Brown University in 1941.

Paul Boschan, trained in electrical engineering and in actuarial work, received the degree of Ph.D. in Vienna in 1934 and was engaged in research work with the Foundation for Visual Education in Vienna and also in actuarial work. Since shortly after his arrival in this country he has been associated with the Institute of Applied Econometrics in New York.

Hans Fried, Ph.D. Vienna (1924), Staatsexamen (1927), was a teacher in Vienna from 1927 to 1938. He spent the year 1939–40 in England. After some months in New York, a half year at the Haverford Workshop and the summer of 1941 at Brown University, he became a research assistant at the Sproul Observatory in the autumn of 1941.

E. J. Gumbel was professor at Heidelberg during the years 1923–33 and at the University of Lyons from 1933 to 1940. He is now living in New York City and is associated with the New School for Social Research.

. *T. Koopmans*, trained as a mathematical physicist and as a mathematical economist, received the doctorate from Leiden in 1936. He lectured at the Commercial University in Rotterdam, and was research associate at the Rotterdam Institute for Economics and economic specialist at the League of Nations. Since 1940, he has lectured at New York University, he has been a research associate at Princeton University and economist with the Penn Mutual Life Insurance Co. (Philadelphia). At present he holds a similar position with the Combined Shipping Adjustment Board in Washington, D. C.

Wolfgang Pauli, professor of theoretical physics in Zürich, 1928–40, has been at the I.A.S. from 1940 on.

Bruno Pontecorvo, physicist, formerly at the University of Rome, has been associated with the Well Surveys Co., at Tulsa, Okla.

Georg Pólya was professor at Zürich from 1928 to 1940. He has spent the two years, 1940–42, at Brown University, with a lectureship at Smith College during a part of the last of these years. He has now been called to a permanent appointment at Stanford University.

Arthur Rosenthal was professor at Munich (1912–22) and at Heidelberg (1922–35) and spent the year 1939–40 in Holland. After a year at the University of Michigan, he has now been appointed to a visiting professorship at the University of New Mexico.

C. L. Siegel was professor at Frankfurt a/M, 1922–37, and at Göttingen, 1938–40. Since his arrival in this country, he has been at the I.A.S.

Heinz Simon, who was a teacher of physics and mathematics in Frankfurt a/M, 1925–39, is now at the Southern Union College (Wadley, Ala.).

Andrew Vasonyi received the degree of Ph.D. at Budapest in 1938 and spent the years 1938–40 in Paris. He was at the Haverford Workshop, 1940–41, at the summer school for advanced mechanics at Brown University, 1941 and 1942, and pursued his studies at Harvard University during the year 1941–42.

Alexander Weinstein has lectured at the Universities of Zürich, Hamburg and Breslau (1928–33). The years 1933–40 were spent in France. After some time in New York City, he received a temporary appointment at the University of Toronto in 1941.

Frantisek Wolf, after having received a doctor's degree at Brno (1928), was a teacher in the schools and a privat-dozent at the Charles University in Prague. The year 1937 was spent in Cambridge and the years 1938–40 in Sweden. A temporary appointment at Macalester College (St. Paul, Minn.) was followed in 1942 by an instructorship at the University of California at Berkeley.

1941

During the year 1941, a number of long delayed immigrations were accomplished, bringing to friendly surroundings some who had been driven from their homes a year or more before. The group included the following:

Ferdinand Beer received the degree of Ph.D. from the University of Geneva in 1937 and had taught in the schools of Paris. He has a position at the Goddard Junior College, in Plainfield, Vt.

L. N. Brillouin, D. Sc., University of Paris, 1920, professor of theoretical physics at the Sorbonne and at the Collège de France, 1928–40, and general director of broadcasting in France, 1939–41, was visiting professor at the University of Wisconsin during 1941–42 (he had been guest lecturer there during the second semester of 1927–28). For the year 1942–43, he has been appointed to the faculty of the school for advanced mechanics at Brown University.

L. Corbeillier, student of the Ecole Polytechnique in Paris (1911–13), holder of a Ph.D. from Paris (1926), as well as of an A.M. in Philosophy, was engaged in engineering work in France and in work at the Ecole Polytechnique. He is now a lecturer on electronics in the Officers Training Course at Harvard.

Max Dehn was professor at Kiel (1911–13), at Breslau (1913–21) and at Frankfurt a/M (1921–35). He spent the year 1939–40 as professor at the Norwegian Technical Institute in Trondheim and came to the United States by way of Siberia. After a professorship of mathematics and philosophy at the University of Idaho, in Pocatello, he was appointed visiting lecturer at Illinois Institute of Technology.

Jacques Hadamard, for many years professor at the Collège de France, is now living in New York City. He lectures occasionally at Columbia University and at neighboring institutions.

Herbert Jehle, D.Eng. from Berlin (1933), collaborator on the *Fortschritte* until 1936, spent the year 1937–38 at Southampton University College, and the years 1938–40 at Brussels. He is now in the physics department of Harvard University.

S. Mandelbrojt, D.Sc. (Paris, 1923), has lectured at the universities of Lille (1928–29) and Clermont-Ferrand (1930–38) and became professor at the Collège de France in 1938. He is now visiting professor at the Rice Institute, where he had also spent the year 1926–27.

Helen Polanyi was a student at the Zürich Technical University in 1915–17. She was at the University of Vienna in 1933–35. At present she is on the faculty of Bennington College.

Willy Prager, D.Eng. (Darmstadt, 1926) has been professor at Darmstadt, Göttingen, Karlsruhe and Istanbul. He is now professor of applied mathematics at Brown University.

Fritz Reiche was professor of theoretical physics in Berlin (1913–21), in Breslau (1921–33) and in Prague (1933–35). He is now living in New York City, associated with the New School for Social Research.

Raphael Salem had training in Paris as an engineer and as a mathematician, receiving the degree of engineer in 1921 and the degree of Sc.D. in 1940. For several years he was engaged in banking. At present he is a lecturer in mathematics at the Massachusetts Institute of Technology.

Hans Samelson, mathematical economist with the degree of D.Sc. from Zürich, is now at the I.A.S.

André Weil, D.Sc. (Paris, 1929) has lectured at Aligarth University (Delhi), 1930–32 and at Strassburg, 1933–40. He spent the year 1941–42 at Haverford College and has now been appointed at Lehigh University.

1942

The list of names is brought to a conclusion by recording the arrival during the first half of 1942 of the following:

E. Kogbetliantz, D.Sc. (Paris, 1923), who lectured in Paris (1921–27), at the University of Teheran (1933–38) and again in Paris (1938–40). He is now joining the faculty of Lehigh University.

Sylvia Nowinska was trained at the University of Lausanne and was for several years assistant at the Paris-Meudon Observatory. She is at present living in New York City.

It has not been possible to give more than very brief indications of the scientific careers, both in this country and abroad, of the mathematicians who have come to this country during the years since 1933. Still less has it been possible to speak of their scientific work, of their plans and hopes, or to have

them record their impressions of the new environment, their prospects and their difficulties. Because this has not been feasible, let me presume to speak for them, even though my arrival in this country falls in a very much earlier period; not however in my own words, but rather in those of Priestley, who replied in the following manner to the congratulatory address presented to him by the American Philosophical Society:

To the Members of the American Philosophical Society of Philadelphia.

GENTLEMEN:—It is with peculiar satisfaction that I receive the congratulations of my brethren of the philosophical Society in this city, on my arrival in this country. It is, in great part, for the sake of pursuing our common studies without molestation, tho', for the present, you will allow, with far less advantage, that I left my native country, and have come to America; and a Society of philosophers, who will have no objections to a person on account of his political or religious sentiments, will be as grateful as it will be new to me.

My past conduct, I hope, will show, that you may depend upon my *zeal* in promoting the valuable objects of your institution, but you must not flatter yourselves, or me, with supposing, that, at any time of life, and with the inconvenience attending a new, and uncertain settlement, I can be of much service to it.

I am confident, however, from what I have already seen of the spirit of the people of this country, that it will soon appear that republican governments, in which every obstruction is removed to the exertions of all kinds of talents, will be far more favourable to Science, and the arts, than any monarchical government has ever been. The patronage to be met with there, is ever capricious, and as often employed to bear down merit as to promote it; having for its real object not science, nor anything useful to mankind, but the mere reputation of the patron, who is seldom any judge of science. Whereas a *Republic* which neither flatters, nor is to be flattered, will not fail in due time to distinguish true merit, and to give every encouragement that is proper to be given in the case. Besides, by opening, as you generously do "an asylum to the persecuted and oppressed of all climes," you will in addition to your own native stock, soon receive a large accession of every kind of merit, philosophical not excepted, whereby you will do yourselves great honour, and secure the most permanent advantage to the community.

J. PRIESTLEY

PHILADA, June 21st 1794.

Perhaps it is in the great scheme of things, that a seed, produced by the distressing events which have occurred during the last ten years, planted in the fertile soil of American science, will grow into a powerful tree, to add beauty and strength to the future life of mankind.

THE APRIL MEETING OF THE OHIO SECTION

The twenty-seventh annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Thursday, April 2, 1942, with an afternoon session, a dinner, and an evening session. Professor Louis Brand, chairman of the Section, presided.

Fifty-six persons registered attendance, including the following forty-five members of the Association: G. E. Albert, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Henry Blumberg, Louis Brand, C. T. Bumer, V. B. Caris, E. H. Clarke, Rufus Crane, Wayne Dancer, O. L. Dustheimer, D. H. Erkiletian, Jr., M. P. Fobes, T. M. Focke, L. R. Ford, R. C. Hildner, Margaret E. Jones, H. W. Kuhn, Lincoln LaPaz, H. W. Linscheid, E. S. Manson, C. G. Maple, Florentina Mathias, Emma J. Olson, C. R. Pettis, Jesse Pierce, H. S. Pollard, Tibor Radó, S. E. Rasor, Maxwell Reade, R. B. Rice, N. S. Risley, K. C. Schraut, H. E. Stelson, C. W. Topp, W. R. Van Voorhis, R. W. Wagner, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, P. M. Young.

The Section was fortunate in having Professor L. R. Ford of Illinois Institute of Technology as the guest speaker. The following officers were elected for the coming year: Chairman, C. T. Bumer, Kenyon College; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, G. E. Albert, Ohio State University; Member of Program Committee, A. C. Ladner, Denison University. It is expected that the next meeting will be held on Thursday, April 1, 1943, at the Ohio State University.

The following six papers were presented:

1. "Non-metric differential invariants" by Professor Louis Brand, University of Cincinnati.

2. "Vector inequalities" by Professor Tibor Radó, Ohio State University.

3. "Mathematical methods in non-linear problems" by Dr. C. A. Ludeke, University of Cincinnati, introduced by the chairman.

4. "Application of tensor analysis to the investigation of geodesic lines on an ellipsoid of revolution" by Professor E. S. Manson, Ohio State University.

5. "The error in using simple interpolation in tables of finance" by Professor H. E. Stelson, Kent State University.

6. "A million ways to solve equations" by Professor L. R. Ford, Illinois Institute of Technology.

In addition to these papers, a portion of the afternoon session was devoted to a symposium in which the members compared experiences as to the effect of the present emergency upon nature and content of courses offered.

Abstracts of some of the papers follow:

1. A relation between two tensor invariants in n-space, generalization of the divergence and curl of three-dimensional vector analysis, and independent of the metric fundamental tensor g_{ij} and the affine connection Γ^k_{ij}, was established by Professor Brand by setting up a one-to-one correspondence between covariant tensors of order $m(0 \leqq m \leqq n)$ and contravariant tensors of order $n-m$,

their duals. It was proposed to define curl P as an absolute, covariant, alternating tensor of order $n-1$, if P is an absolute covariant tensor of order $n-2$; while rot P, defined as the dual of curl P, is a contravariant vector density.

2. An inequality such as the inequality of Schwartz is easily proved while others are more complicated in appearance and proof. For many of these latter, simple proofs can be given based on theorems concerning the area of a triangle in terms of various data. One must first recognize a given inequality as a special case of a more general inequality involving vectors. Professor Radó presented several examples and suggested that such applications of theorems be used as stimulating exercises even though the inequalities derived be beyond the scope of instruction.

3. Dr. Ludeke presented four methods used in solving certain types of non-linear differential equations. The first, known as the method of isoclines, and applicable to non-linear equations of the form $dv/dx=\beta(x)\,v^{n-1}+k(x)/v=f(x, v)$, where $v=dx/dt$ and $f(x, v)$ is single valued and continuous: this method obtains a first integral between x and v by plotting the family of curves $f(x, v)=$constant, and attaching to them the corresponding slope of the integral curve. The integral between x and t is determined by a second graphical integration. A second method, applicable to the preceding equation with $\beta(x)=0$, is used to find only the frequency of the solution by direct integration over a quarter cycle in steps small enough to ensure approximate linearity of $k(x)$. A third method, used in solving the equation $md^2x/dt^2+\beta_n(dx/dt)^n+kx=P\sin\alpha t$, replaces the term $\pm\beta_n(dx/dt)^n$ by an equivalent term $\beta_1(dx/dt)$; the equivalence being determined by the dissipative work of each per cycle. The equation is then linear and the solution is well known. A fourth method, applicable to equations of the form $md^2x/dt^2+k(x)=P\sin\alpha t$, assumes the solution in the form $x=\sum_{n=1,3,5,\cdots}^{\infty} a_n \sin n\alpha t$, and $k(x)$ in the form $\sum_{n=1,3,5,\cdots}^{\infty} A_n\sin n\alpha t$; and then proceeds to determine the a_i so that when the A_i are calculated from them the series $\sum_{n=1,3,5\cdots}^{\infty} A_n \sin n\alpha t$ gives the correct form $k(x)$.

4. Professor Manson discussed the application of the equations of a geodesic obtained by tensor analysis,

$$d^2t_\alpha/ds^2+\left\{\mu\nu,\ \alpha\right\}\frac{dx_\mu}{ds}\,\frac{dt_\nu}{ds}=0$$

to the two dimensional case of the geodesic on the surface of a spheroid of revolution. Equations were developed showing the relation between the longitude θ, and the latitude ϕ, and also between the distance s, and ϕ. Professor Manson showed how these equations may be used in computing the distance along a geodesic between two points of known latitude and longitude.

5. Formulas for determining the error in using simple interpolation in tables of finance were derived by Professor Stelson. The cases considered were interpolation for the time, rate, and amount. Algebraic expressions were derived showing the maximum error and the average error in each case. Approximate rules for the accuracy of interpolation were presented.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association in Oklahoma City on Friday morning, February 13, 1942. Professor C. E. Springer, chairman of the Section, presided.

Forty-seven representatives of high schools and colleges attended the meeting, including the following sixteen members of the Association: E. F. Allen, Joseph Barnett, Jr., J. C. Brixey, N. A. Court, A. H. Diamond, H. L. Hall, O. H. Hamilton, J. O. Hassler, E. E. Heimann, J. E. LaFon, Dora McFarland, W. C. Randels, S. W. Reaves, W. T. Short, C. E. Springer, B. S. Whitney.

At the business session the following officers were elected: Chairman, O. H. Hamilton, Oklahoma A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following six papers:

1. "The bisectral center of four spheres" by Professor N. A. Court, University of Oklahoma.

2. "Equi-continuous collections of continuous transformations" by Professor O. H. Hamilton, Oklahoma A. and M. College.

3. "A comparison of the effectiveness of mathematics teaching in different types of Oklahoma colleges" by Professor J. O. Hassler.

4. "Conformal mapping of contours in the study of potential flow" by Professor A. H. Diamond, Oklahoma A. and M. College.

5. "On new methods in differential equations with applications to the structural analysis of airplanes" by Professor Stefan Bergmann, Brown University, introduced by the Secretary.

6. "Quadratic forms which represent all integers, except those in certain geometric progressions" by Professor S. B. Townes, University of Oklahoma, introduced by Professor Springer.

Abstracts of the papers follow.

1. Professor Court defined the "sect" v^2 of a point V for a sphere (A, a) as the square of the distance VA of the point V from the center A of (A, a) increased by the square of the radius a of (A, a). The sphere (V, v) having V for center and v for radius bisects the sphere (A, a). He pointed out the degree of parallelism between "sect" and "power." He showed that, given four spheres (A), (B), (C), (D), there is a sphere (V) bisecting those spheres, and that the sphere (V), the orthogonal sphere (U) of the given spheres, and the sphere $(O)=ABCD$ determined by the centers of the given spheres, form a coaxal pencil. The sphere (V) also bisects the sphere having for center the centroid G of the tetrahedron $(T)=ABCD$ and for the square of its radius one-fourth of the sum of the squares of the radii of the given spheres increased by one-sixteenth of the sum of the squares of the edges of (T).

2. Professor Hamilton discussed the definitions of a continuous transformation on a set of points, homeomorphism, and an equi-continuous collection of

homeomorphisms. He gave a proof of the theorem: If T is a homeomorphism of a compact continuum N, lying in a Euclidean space of n dimensions, into itself; and if the integral powers of T form an equicontinuous collection of homeomorphisms, then there can be defined on N a metric function, $D(A, B)$, which is left invariant by the homeomorphism T.

3. Professor Hassler reported the results of an investigation of the grades in mathematics made at the University of Oklahoma by students transferring there from other colleges in the state. He classified the students according to the nature of the colleges from which they came, and made a comparison of the effectiveness of the teaching of mathematics in the different types of colleges.

4. The study of potential flow about biplane wing sections in a two-dimensional stream leads to the problem of conformal mapping of a doubly connected domain into a ring. The plane in which the doubly connected domain lies was first mapped by Professor Diamond into the interior of a rectangle by means of a function defined by an elliptical integral of the first kind. Finally he mapped the rectangle into a ring by the exponential function.

5. Professor Bergmann discussed the mathematics involved in the structural analysis of airplanes.

6. Professor Townes discussed forms $ax^2+by^2+cz^2+dw^2$ which represent all integers except those in certain geometrical progressions. For example, $x^2+y^2+7z^2+7w^2$ was found to represent all integers except $7^k \cdot 3$ and $7^k \cdot 6$, $k \geqq 0$.

J. C. BRIXEY, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The nineteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Heidelberg Hotel in Jackson, Mississippi, on Friday and Saturday, March 6–7, 1942.

The attendance was about sixty-five, including the following twenty-eight members of the Association: P. H. Anderson, T. A. Bickerstaff, Leora Blair, W. H. Bradford, H. E. Buchanan, R. L. Coker, W. E. Cox, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, F. C. Gentry, H. S. Kaltenborn, H. T. Karnes, C. G. Killen, A. C. Maddox, Dorothy McCoy, B. E. Mitchell, I. C. Nichols, W. V. Parker, H. L. Quarles, S. T. Sanders, Jr., H. F. Schroeder, C. D. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, J. A. Ward, F. L. Wren.

Sessions were held Friday afternoon and evening and Saturday morning. Professor B. A. Tucker, chairman of the Section, presided at all sessions. The annual dinner was held on Friday evening. The visiting speaker, Professor F. L. Wren of George Peabody College for Teachers, delivered an address at the dinner and also spoke at the Saturday morning session.

At the business session on Saturday morning it was decided to hold the meeting at Delta State Teachers College, Cleveland, Mississippi, in 1944. The fol-

lowing officers were elected for 1942–43: Chairman, J. A. Ward, Delta State Teachers College; Vice-Chairman for Louisiana, S. T. Sanders, Jr., Southwestern Louisiana Institute; Vice-Chairman for Mississippi, B. O. Van Hook, Millsaps College; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute.

The following ten papers were presented:

1. "A solid of revolution" by Professor J. A. Ward, Delta State Teachers College.

2. "Poristic polygons" by Professor H. E. Buchanan, Tulane University.

3. "Fire insurance, a one-, a three-, or a five-year policy?" by Professor I. C. Nichols, Louisiana State University.

4. "The problem of Apollonius" by Mrs. Alta H. Samuels, Hinds Junior College, introduced by the Secretary.

5. "The twelve squares ascribed to a triangle" by Professor B. E. Mitchell, Millsaps College.

6. "A note on Heaviside operators" by Professor J. F. Thomson, Tulane University.

7. "Distributions in stratified sampling" by Dr. P. H. Anderson, Louisiana State University.

8. "Boundary values for probabilities in problems of two variables" by Professor C. D. Smith, Mississippi State College.

9. "The next step forward" by Professor F. L. Wren, George Peabody College for Teachers.

10. "A contraction method for determinant expansion" by Professor F. L. Wren, George Peabody College for Teachers.

Abstracts of some of these papers follow:

1. Professor Ward showed how to obtain the median curve of the solid of revolution obtained by rotating a rectangular parallelopiped about a diagonal. He showed that there are eight different types of median curves, depending on the ratios of the edges, and displayed graphs of several of them.

2. After reproducing a part of Jacobi's paper in order to get an intelligible start Professor Buchanan proved the following theorems: (1) The diagonals of the quadrilateral formed by the points of tangency of a poristic polygon are mutually perpendicular, and intersect at a fixed point on the line of centers of the two circles; (2) The external diagonal of a poristic quadrilateral is a fixed line perpendicular to the line of centers; (3) Every poristic quadrilateral determines another poristic quadrilateral which is inscribed in the smaller of the original circles and circumscribed about a third circle whose center can be found by a geometric construction.

3. Using the assumptions first, that insurance is paid in advance; second, that discounts of half a year and of one whole year are given on three-year and five-year policies respectively; and, third, that the fire rate itself remains unchanged for the duration of the periods under discussion, Professor Nichols showed that (1) a one-year policy is consistently higher than a three-year policy.

(2) A three-year policy is cheaper than a five-year policy for a rate of interest of 4% or higher, the difference becoming greater as the rate increases. (3) For a rate of interest of $3\frac{1}{2}\%$ or less the five-year policy is cheaper than the three-year policy, the rate at which these two policies are equal being slightly under 4%. Therefore, in general practice, the three-year policy should be written.

4. Mrs. Samuels showed that nine of the ten cases of the problem of Apollonius may be reduced by inversion to three cases, namely, *PPL*, *PCC*, and *CCC*. She then solved these three cases by inversion, emphasizing their simplicity. In each solution, through the mechanics of inversion, three points through which the required circle must pass are easily located, thereby reducing each problem to *PPP*, the first and simplest of the ten cases of the problem of Apollonius.

7. Dr. Anderson compared the random sample distribution of a statistic with the stratified sample distribution of the same statistic. The statistics considered were the mean, the standard deviation, and the student's ratio. The samples were taken from a rectangular, a normal, a skewed, and a general population. This paper appeared in the *Annals of Mathematical Statistics*, March 1942.

8. Beginning with the fundamental concept of Bernoulli, Professor Smith compared five types of probability. Consider the joint occurrence of two characteristics of an item $U_{i,j}$. If the characteristics fall in column (i) and row (j) of the xy-plane and $M_n(j, i)$ is the moment about the y-axis, $P_j \leqq \mu_n(j, i)/\lambda^n$, where P_j is the probability that the item $U_{i,j}$ falls farther from the x-axis than a distance λ. A similar bound with respect to the y-axis gives a rectangle within which $U_{i,j}$ does not fall. In case of correlated characteristics a boundary parallelogram was found. The bounds were reduced by cases where the frequency does not increase with λ. For frequency surfaces, a boundary circle and a boundary ellipse were given.

9. Professor Wren raised and discussed the following questions: What is the next step forward in the mathematics program in the elementary and secondary schools of the United States? The present emergency has given a great deal of emphasis to the importance of mathematics. It has also thrown the spotlight on some of the weaknesses of the mathematics program of the past and present. As we project our thinking into the forthcoming peace era, what are the implications as to curriculum, methods of teaching, and preparation of teachers? What should the Association, Council, and Society be doing now in the planning of this peace-time program for mathematics?

10. Professor Wren proved the theorem which establishes the contraction method for the evaluation of determinants. He then applied this technique to find the quotient of two determinants, to solve systems of simultaneous linear equations, to find the common root of two equations of the form $a_0X^3+a_1X^2+a_2X+a_3=0$ and $b_0X^2+b_1X+b_2=0$, and to find the moments of the point binomial $(p+q)^n$.

W. V. PARKER, *Secretary*

THE TWENTY-EIGHTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-eighth meeting of the Kansas Section of the Mathematical Association of America was held at Fort Hays Kansas State College, Hays, on Friday and Saturday, March 27–28, 1942, in conjunction with the meetings of the Kansas Academy of Science and the Kansas Association of Teachers of Mathematics. There were three sessions at all of which Professor C. F. Lewis, vice-chairman of the Section, presided. Professor C. V. Bertsch, elected chairman a year ago, is now connected with American University, Washington, D. C.

There were forty in attendance, including the following twenty-five members of the Association: Wealthy Babcock, Florence L. Black, B. H. Buikstra, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, Paul Eberhart, F. D. Faulkner, W. H. Garrett, Edison Greer, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, P. S. Pretz, G. B. Price, C. B. Read, J. A. G. Shirk, D. T. Sigley, G. W. Smith, W. T. Stratton, Gilbert Ulmer, E. B. Wedel, A. E. White.

The officers elected for the coming year are: Chairman, C. F. Lewis, Kansas State College, Manhattan; Vice-Chairman, Paul Eberhart, Washburn Municipal University of Topeka; Secretary, Anna Marm, Bethany College. The decision on the time and place for the next meeting was left to the executive committee.

The Committee on the Third Placement Test, given at the beginning of the 1941–42 school year to Freshmen in most of the colleges and junior colleges of the state, reported through the chairman, Professor Gilbert Ulmer, and Professor O. J. Peterson. There was general discussion on the test in which many of those present took part. The Committee was asked to continue the Test work in 1942–43.

Professor Emma Hyde gave a tribute to the memory of Professor U. G. Mitchell, expressing the feeling in all our hearts, as we met for the first time without his presence—thankfulness for the years during which he has been our leader and our inspiration, and sadness for the parting now.

The following eight papers and reports were presented:

1. "Continued fractions of quaternions" by E. G. Swafford, Fort Hays, Kansas State College, introduced by Professor Colyer.

2. "A system of linear differential equations with a regular singular point" by F. D. Faulkner, Kansas State College.

3. "Some properties of Sine Z as the inverse of an integral" by B. H. Buikstra, Kansas State College.

4. "Some formulas in analytic geometry" by Professor G. B. Price, University of Kansas.

5. "Mathematics used by the Armed Forces" by Professor D. T. Sigley, Kansas State College.

6. "Some circles related to the triangle" by Professor G. W. Smith, University of Kansas.

7. "Is mathematics an exact science?" by Professor C. B. Read, University of Wichita.

8. "Report of the committee on the Placement Test" by Dean Gilbert Ulmer and Professor O. J. Peterson.

Abstracts of most of the papers follow.

1. Mr. Swafford showed that any right-handed quotient of two quaternions of the Hurwitz integral domain H can be represented by a finite regular right-handed continued fraction of quaternions of H by using an euclidean algorithm to determine a greatest common right divisor of the two quaternions.

2. Mr. Faulkner discussed in detail the solutions of the system of differential equations,

$$t(dX_1/dt) = \theta_{11}(t)X_1 + \theta_{12}(t)X_2,$$
$$t(dX_2/dt) = \theta_{21}(t)X_1 + \theta_{22}(t)X_2,$$

where the θ's are functions of t holomorphic at the origin, at least one of which is not zero at the origin.

4. Professor Price considered the formulas for lengths, areas, volumes, . . . expressed in terms of certain determinants of the coördinates of the points, and showed how they followed a simple and definite pattern thus explaining to an analytic geometry class how mathematics grows by generalization. A second set of formulas, related to the above, gives the distance from a point, line, plane, . . . determined by one point, two points, three points, . . . in m-dimensional euclidean space.

5. Professor Sigley presented some observations on the status of mathematical development in the United States at the present time, and on the relation of this development to the war. He enumerated typical problems, problems that men in the armed forces and related industries must solve, classified according to mathematical principle involved. It seems that present offerings in mathematical curricula are sufficient for proper training of the personnel of the armed forces. More time could be devoted profitably to these subjects, with additional emphasis placed on the study of charts, graphs, tables, interpolation, and other phases of graphical and numerical methods.

6. By means of a set of charts, Professor Smith discussed the geometry of a plane triangle; especially the circles that are associated with a triangle, such as the circumscribed, the inscribed, and the escribed circles, the inexcircles, the exexcircles, the nine-point circle, the isogonic circles, the Spieker circle, the circles of Droz-Farney, the Fuhrmann and the Brocard circles, several of the Tucker circles, the circles of Apollonius, the Canon circles and the Blanc circles. Although the triangle has been studied for a long time and by a great many investigators, it still presents many problems, several of which Professor Smith pointed out.

7. Professor Read called attention to several instances in which texts and reference books in mathematics contain definitions or statements which are contradictory. The plea was made that, if there is a disagreement as to definition of terms, the author might at least call attention to the fact that there is variation in usage. Among illustrations used, were: the meaning of factorial n for

fractional values of n; definition of the principal values of the inverse trigonometric functions; inconsistencies in number classification; confusion in the use of terms *total force* and *total pressure*; definitions of an asymptote; use of the term *rate of change*; definition of a mantissa. The last three mentioned have been discussed in the "Questions, Discussions and Notes" department of this MONTHLY, October 1939, December 1939, and March 1941.

LUCY T. DOUGHERTY, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-first regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa Wesleyan College, Mount Pleasant, Iowa, on Friday and Saturday, April 17–18, 1942, in conjunction with the fifty-sixth annual meeting of the Iowa Academy of Science. Dean O. C. Kreider, chairman of the Section presided. He was relieved by Professor Fred Robertson for part of the session on Saturday morning.

The attendance was about twenty-six including the following seventeen members of the Association: J. W. Beach, J. O. Chellevold, L. M. Coffin, N. B. Conkwright, A. W. Davis, C. W. Emmons, Cornelius Gouwens, L. A. Knowler, O. C. Kreider, R. B. McClenon, F. M. McGaw, E. E. Moots, H. V. Price, Fred Robertson, W. J. Rusk, E. R. Smith, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the section elected for 1942–1943 are as follows: Chairman, N. B. Conkwright, State University of Iowa; Vice-Chairman, H. E. Ellingson, Luther College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the host, Iowa Wesleyan College, and by Professor G. E. King of the department of mathematics, was adopted at the business meeting.

The following resolution on the closing of the earthly career of Miss Maria M. Roberts of Iowa State College was presented and passed:

Miss Maria M. Roberts entered Iowa State College as a student in the Fall of 1887, and upon her graduation was retained on the staff of the department of mathematics as instructor. Her student career was so satisfactory and notable that this was a worthy recognition of her abilities. Thereafter for more than fifty years she was connected with the college as teacher or as administrator in connection with the Junior College.

Miss Roberts was a true teacher, inspiring guide, and helpful friend to the host of youth who had the privilege of coming under her influence. As co-author with Miss Julia Colpitts, Miss Roberts made in their text in analytical geometry a valuable contribution to the teacher's tools. The Association has lost a faithful and helpful member, and Iowa State College a worthy alumna and teacher. A

large number of former students everywhere will join in spirit in this resolution of memory and esteem.

The following thirteen papers were read on the Association program, the last four by title:

1. "Summation of certain series of constants by the method of residues" by Professor J. J. L. Hinrichsen, Iowa State College, introduced by the Secretary.

2. "The number of isomeric alkenes" by R. E. Carr, Iowa State College, introduced by the Secretary.

3. "The adequacy of the frequency definition of probability" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

4. "Trends in teaching" by Professor P. G. Robinson, Iowa State College, introduced by the Secretary.

5. "Spherical trigonometry as an emergency course" by Professor R. B. McClenon, Grinnell College.

6. "Survey courses" by Dean O. C. Kreider, Ellsworth Junior College.

7. "Report and discussion of survey courses by the Section Committee" by Professor Fred Robertson, Iowa State College, Chairman.

8. "Report of the representative on the Board of Governors for Region Number 9, Iowa—Minnesota—Wisconsin" by Professor Cornelius Gouwens, Iowa State College.

9. "Systems of linear equations with coefficients subject to error" by Dr. A. T. Lonseth, Iowa State College. (Read by Professor Gouwens.)

10. "Boolean matrices" by Professor E. W. Chittenden, State University of Iowa. (Read by title.)

11. "A test for the nature of the roots of a cubic equation" by Professor E. E. Watson, introduced by the Secretary. (Read by title.)

12. "The asymptotic distribution of means of stratified samples" by Professor A. T. Craig, Iowa State University. (Read by title.)

13. "Gamma function expansions" by Professor Henry Van Engen, State Teachers College. (Read by title.)

Abstracts of some of the papers follow:

1. Professor Hinrichsen reviewed the theory of residues and its application to the summation of certain convergent series of real constants. This method was illustrated by summing the series $\sum_{n=1}^{\infty}(1/1+n^2)$ and more generally $\sum_{n=1}^{\infty}(1/1+n^{2p})$, where p is a positive integer. The usefulness of the method is restricted in that the nth term of the series must be an even function of n.

2. Mr. Carr described a method for finding the number of isomeric alkenes. The alkenes are considered to be built up by attaching four alkyl radicals to the four available bonds of a basic skeleton

The alkenes are classified in 30 types, chosen according to the degree of similarity

between the attached alkyl groups. Formulas developed by Mr. Carr for each type enable one to compute the numbers of different varieties of isomers.

3. Recently several writers have named instances of probabilities of isolated events—probabilities, according to them, incapable of description in terms of frequency limits. Professor Allen, analyzing them, held that every numerical probability can be regarded as the limit of relative frequency of success in actual or imagined sequences of trials.

4. Dr. Robinson gave a refreshing discussion on trends in teaching, particularly in the direction of thinking on the part of the student as contrasted with the emphases on skills.

5. Professor McClenon gave an outline of the content of a fundamental course in spherical trigonometry and exhibited interesting analogies between formulas in spherical trigonometry and in plane trigonometry.

6. Dean Kreider stated that the various survey courses may be classified as of the review type, the eclectic, the analytical, the cultural, the historical or the psychological. The main characteristics of survey mathematics should be breadth, psychological organization, and value. The keynote should be to make mathematics really and obviously worth while to students and to encourage students rather than to eliminate them. This paper will appear in the *Proceedings of the Iowa Academy*.

7. Professor Robertson offered evidence showing the need for survey courses, described types which have been offered and exhibited recent texts in the field. An interesting discussion developed.

8. Dr. Gouwens presented a report on the state of the Mathematical Association of America and an account of the business meetings of the Board of Governors during the past year.

9. Dr. Lonseth presented formulas for maximum errors in the solutions of a system of n linear equations in n unknowns where the coefficients are subject to small errors.

10. Professor Chittenden discussed the algebra of matrices whose elements belong to a Boolean algebra. If the Boolean algebra is finite, the resulting algebra of matrices is an interesting form of non-commutative distributive algebra. Tables of incidence relations in topology are well known examples of such matrices in case the basic Boolean algebra has only two elements. He defined addition and multiplication of such matrices in the usual manner and rank in terms of linear dependence.

CORNELIUS GOUWENS, *Secretary*

REMARKS ON THE ABEL-DINI THEOREM

T. H. HILDEBRANDT, University of Michigan

1. The Abel-Dini Theorem* may be stated as follows: If d_n is a sequence of positive numbers such that the infinite series $\sum_n d_n$ is divergent and if $s_n = \sum_{m=1}^{n} d_m$, then the infinite series $\sum_n \frac{d_n}{s_n^{1+\alpha}}$ is convergent for $\alpha > 0$, and divergent for $\alpha \leqq 0$. A corresponding result holds for convergent series, excepting that the remainders replace the sums, viz., if c_n are positive numbers such that $\sum_n c_n$ is convergent and if $r_n = \sum_{m=n+1}^{\infty} c_m$, then $\sum_n \frac{c_n}{r_{n-1}^{1+\alpha}}$ is convergent for $\alpha < 0$ and divergent for $\alpha \geqq 0$. Of the two theorems, the second is slightly easier to demonstrate. For the divergence feature it is sufficient to consider the case where $\alpha = 0$, make the substitution $c_n = r_{n-1} - r_n$ and make the observation that

$$\sum_{n=k}^{m} \frac{c_n}{r_{n-1}} = \sum_{n=k}^{m} \frac{r_{n-1} - r_n}{r_{n-1}} > \sum_{n=k}^{m} \frac{r_{n-1} - r_n}{r_{k-1}} = 1 - \frac{r_m}{r_{k-1}}$$

which approaches 1 in m for fixed k. For the convergence theorem we take $\alpha = -\beta$, with $0 < \beta < 1$ and note that

$$\frac{c_n}{r_{n-1}^{1+\alpha}} = \frac{r_{n-1} - r_n}{r_{n-1}^{1-\beta}} = r_{n-1}^{\beta}\left[1 - \frac{r_n}{r_{n-1}}\right] \leqq \frac{1}{\beta} r_{n-1}^{\beta}\left[1 - \left(\frac{r_n}{r_{n-1}}\right)^{\beta}\right]$$

$$= \frac{1}{\beta}\left[r_{n-1}^{\beta} - r_n^{\beta}\right]$$

since $x^{\beta} \leqq 1 + \beta(x-1)$ for $0 < \beta < 1$ and $x \geqq 0$. Then since $\sum_n (r_{n-1}^{\beta} - r_n^{\beta})$ converges it follows that $\sum_n c_n / r_n^{1-\beta}$ converges for $0 < \beta < 1$ and consequently for $\beta > 0$. The Abel-Dini theorem, where $\sum_n d_n$ diverges, now follows by making the substitution $r_n = \frac{1}{s_n}$ so that $c_n = \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{d_n}{s_{n-1} s_n}$. Then $\frac{c_n}{r_{n-1}^{1+\alpha}} = \frac{d_n}{s_{n-1}^{-\alpha} s_n}$. It is interesting to note, however, that the convergence part of the theorem now yields the Pringsheim modification of the Abel-Dini theorem, viz., if $\sum_n d_n$ is divergent then $\sum_n \frac{d_n}{s_{n-1}^{\alpha} s_n}$ is convergent for $\alpha > 0$.† Obviously one can get a "Pringsheim" modification for $\sum_n c_n$ convergent by starting with the Abel-Dini theorem and setting $s_n = \frac{1}{r_n}$.

* Cf., for instance, Knopp, Infinite Series, 1928, pp. 290–293.

† Cf. Knopp: loc. cit., p. 300.

If in the proof that $\sum \frac{c_n}{r_{n-1}^{1-\beta}}$ is convergent for $\beta>0$, we should consider instead $\frac{c_n}{r_n^{1-\beta}}$, we find that the inequality $x^\beta-1\leqq\beta[x^\beta-1]$ would need to be applied in the reverse form, yielding nothing. It raises the question whether the theorem is valid in this form. The answer is in the negative.

For this purpose we take $r_n=1/n^{n^n}=n^{-n^n}$. Then $c_n=(n-1)^{-(n-1)^{n-1}}-n^{-n^n}$ and $c_n/r_n^{1-\beta}=(n-1)^{-(n-1)^{n-1}}\cdot n^{(1-\beta)n^n}-n^{-\beta n^n}$, the last term of which obviously approaches zero, but the first term of which for $0<\beta<1$ is obviously greater than $(n-1)^{(1-\beta)n^n-(n-1)^{n-1}}$ which approaches infinity for all $0<\beta<1$; i.e., it is possible for $\sum_n c_n/r_n^\alpha$ to diverge for all $\alpha>0$. We might note, however, that if $r_n=1/n^n$ then $\sum_n c_n/r_n^{1-\beta}$ converges for $\beta>0$. Similar results hold for the divergent series, *i.e.*, there exist divergent series $\sum_n d_n$ such that $\sum_n d_n/s_{n-1}^\alpha$ is divergent for all $\alpha\geqq 0$.

2. The Abel-Dini theorem gives at once the properties of the harmonic series $1/n^{1+\alpha}$ by taking $d_n=1$; as a matter of fact it might be considered a generalization of this. If we in turn apply the Abel-Dini theorem to $d_n=1/n$ we get the statement: $\sum \frac{1}{n\left(\sum_1^n \frac{1}{m}\right)^{1+\alpha}}$ is convergent if $\alpha>0$ and divergent if $\alpha\leqq 0$. Since $\lim_n \frac{\log n}{\sum_1^n \frac{1}{m}}=1>0$, this is equivalent to $\sum_n \frac{1}{n\log^{1+\alpha} n}$ is convergent if $\alpha>0$ and divergent if $\alpha\leqq 0$. The next steps $\sum_n \frac{1}{n\log n(\log\log n)^\alpha}$ and so on, seem to require more complicated consideration.* These results suggest the question whether it is possible to prove similar theorems about $\sum_n \frac{d_n}{s_n(\log s_n)^{1+\alpha}}$, $\sum_n \frac{d_n}{s_n\log s_n(\log\log s_n)^{1+\alpha}}$ and so on.

We note that since $s_n\to\infty$ implies $\log s_n\to\infty$, $\log s_n-\log s_{n-1}$ are the terms of a divergent series. Hence $\sum_n \frac{\log s_n-\log s_{n-1}}{(\log s_n)^{1+\alpha}}$ with $s_0=1$ is convergent if $\alpha>0$ and divergent if $\alpha\leqq 0$. Now the series $\sum_n \frac{d_n}{s_n(\log s_n)^{1+\alpha}}$ will have the same convergence and divergence properties as $\sum_n \frac{\log s_n-\log s_{n-1}}{(\log s_n)^{1+\alpha}}$ if the ratio of the corresponding terms is bounded from zero and infinity. This ratio is equal to

$$\frac{d_n}{s_n(\log s_n-\log s_{n-1})}=\frac{1-s_{n-1}/s_n}{\log(s_n/s_{n-1})}.$$

* Cf., however, Knopp, loc. cit., pp. 292–293.

Now the function $\dfrac{x-1}{\log x}$ for $0<x<1$ increases monotonically from 0 to 1. Hence we have the theorem:

THEOREM I. *If* $s_n=\sum_{m=1}^{n}d_m$ *where* d_n *are positive and the terms of a divergent series and if* s_{n-1}/s_n *is bounded from zero in* n *(or* d_n/s_n *is bounded from unity), then* $\sum_n \dfrac{d_n}{s_n(\log s_n)^{1+\alpha}}$ *is convergent for* $\alpha>0$, *and divergent for* $\alpha\leqq 0$.

Since for the convergences of a series $\sum_n u_n$ it is sufficient that the ratio of the terms u_n/a_n with $\sum_n a_n$ convergent be bounded we can assert:

THEOREM Ia. *If* $d_n>0$ *are the terms of a divergent series and* $s_n=\sum_{m=1}^{n}d_m$, *then* $\sum_n \dfrac{d_n}{s_n(\log s_n)^{1+\alpha}}$ *is convergent for* $\alpha>0$.

However, the divergency for $\alpha\leqq 0$ cannot be asserted for all d_n, particularly if $\lim d_n/s_n=1$. We note the following instances. If $s_n=n^n$, then $s_{n-1}/s_n\to 0$. However, $\sum_n d_n/s_n(\log s_n)^{1+\alpha}=\sum_n \dfrac{n^n-(n-1)^{n-1}}{n^n}\dfrac{1}{n^{1+\alpha}(\log n)^{1+\alpha}}$ is obviously divergent for $\alpha\leqq 0$. On the other hand, if $s_n=n^{n^n}$, then $d_n/s_n(\log s_n)^{1+\alpha}$ is less than $\dfrac{1}{n^{(1+\alpha)n}\log^{1+\alpha} n}$ and consequently gives rise to a convergent series for $\alpha>-1$.

The following extension to iterated logarithms holds:

THEOREM II. *If* $d_n>0$ *are the terms of a divergent series, and* $s_n=\sum_{m=1}^{n}d_m$ *and if* s_{n-1}/s_n *is bounded from zero (or* d_n/s_n *from unity), then*

$$\sum_n \frac{d_n}{s_n \log s_n (\log\log s_n)^{1+\alpha}}$$

is convergent for $\alpha>0$ *and divergent for* $\alpha\leqq 0$.

By the Abel-Dini theorem, summing from a value of n_0 so that $\log s_n\neq 0$ and $\log\log s_n\neq 0$ for $n\geqq n_0$, the series $\sum_n \dfrac{\log\log s_n-\log\log s_{n-1}}{(\log\log s_n)^{1+\alpha}}$ will be convergent for $\alpha>0$ and divergent for $\alpha\leqq 0$. The ratio of the terms of this series and of the series of the theorem can be transformed as follows:

$$\frac{d_n}{s_n\log s_n\log(\log s_n/\log s_{n-1})}=\frac{s_n-s_{n-1}}{s_n[\log s_n-\log s_{n-1}]}\cdot\frac{1-\log s_{n-1}/\log s_n}{\log(\log s_n/\log s_{n-1})}$$

$$=\frac{1-s_{n-1}/s_n}{\log(s_n/s_{n-1})}\cdot\frac{1-\log s_{n-1}/\log s_n}{\log(\log s_n/\log s_{n-1})}.$$

Since s_{n-1}/s_n is bounded from zero, the first term of the product is bounded from zero and infinity. Further, if s_{n-1}/s_n is bounded from zero, then

$$\frac{\log s_{n-1}}{\log s_n} = 1 + \frac{\log (s_{n-1}/s_n)}{\log s_n}$$

obviously approaches 1 as n and s_n approach infinity. Hence the product of these terms is bounded from zero and infinity. Obviously we can assert the convergence of $\sum_n \frac{d_n}{s_n \log s_n (\log \log s_n)^{1+\alpha}}$ for any $\alpha > 0$.

One might inquire whether the divergence of $\sum_n \frac{d_n}{s_n \log s_n}$ with $\frac{s_{n-1}}{s_n}$ approaching zero is sufficient to guarantee the divergence of $\sum_n \frac{d_n}{s_n \log s_n (\log \log s_n)}$. The example $s_n = n^n$ gives

$$\frac{d_n}{s_n \log s_n \log \log s_n} = \frac{n^n - (n-1)^{n-1}}{n^n} \cdot \frac{1}{n[\log^2 n + (\log n) \log \log n]}$$

and shows that the reply is negative.

Obviously the reasoning of Theorem II can be continued yielding the result: If the ratio s_{n-1}/s_n is bounded from zero, then the series

$$\sum_n \frac{d_n}{s_n \log s_n \cdots [(\log)^k s_n]^{1+\alpha}} \quad \text{where } (\log)^k s_n = \underbrace{\log \log \cdots \log}_{k \text{ times}} s_n$$

is convergent if $\alpha > 0$ and divergent if $\alpha \leqq 0$, the series starting with an n_0 such that for $n \geqq n_0$ and $k \geqq 1$, $(\log)^k s_n$ does not vanish.

3. The harmonic series $a_n = 1/n$ with the property that $\sum_n a_n^{1+\alpha}$ is a convergent series for $\alpha > 0$ and divergent series for $\alpha \leqq 0$ suggests the question: Do similar properties hold if $a_n = d_n/s_n$, $d_n > 0$, $\sum_n d_n$ divergent? No general theorems can be proved because:

If $D_n > 0$ is any series such that $D_1 = 1$ and $D_n < 1$, $n > 1$, and $\sum_n D_n$ is divergent, then there exists another divergent series $\sum_n d_n$ such that $d_n/s_n = D_n$. For $d_n/s_n = (s_n - s_{n-1})/s_n = D_n$ gives

$$s_n = \frac{s_{n-1}}{1 - D_n}.$$

Then

$$s_n = \frac{s_1}{\prod_{m=1}^{n} (1 - D_m)}.$$

Since $s_n > s_{n-1}$ we obtain at once $d_n > 0$, in terms of $s_1 = d_1$. Obviously $s_n \to \infty$.

In the case of the harmonic series $a_n = 1/n$, the number 1 is the dividing line between the numbers α such that $\sum a_n^\alpha$ converges and diverges, 1 being on

the divergent side. Do there exist series where the dividing number is on the convergent side; *i.e.*, do there exist sequences $a_n>0$ such that $\sum_n a_n^\alpha$ is convergent for $\alpha\geqq\alpha_0$ and divergent for $\alpha<\alpha_0$? Obviously if the statement is true for $\alpha_0=1$, it can be obtained for any α_0 replacing a_n by a_n^{1/α_0}. For $\alpha=1$ we have the instance $a_n=\dfrac{1}{n\log^{1+e}n}$ where $e>0$. Obviously $\sum_n a_n$ is convergent and so $\sum_n a_n^\alpha$, $\alpha>1$, will also be convergent. On the other hand, $\sum_n \dfrac{1}{[n\log^{1+e}n]^{1-\alpha}}$ is divergent for $\alpha>0$. This follows by comparison with $\dfrac{1}{n^{1-\beta}}$, $0<\beta<\alpha$ and the fact that $\lim_{x\to\infty}\dfrac{x^e}{\log x}=\infty$ if $e>0$. A similar trick enables us to prove the following theorem of which the result just mentioned is a special case:

THEOREM III. *If $d_n>0$ is such that $\sum_n d_n$ is divergent, then there exists a sequence c_n such that $0<c_n\leqq d_n$ for all n, $\sum_n c_n$ is convergent but $\sum c_n^\alpha$ is divergent for $\alpha<1$.*

By way of proof we let $D_n=d_n$ if $d_n\leqq 1$, and $D_n=1$ if $d_n\geqq 1$. Let $S_n=\sum_{m=1}^n D_m$. Then obviously $S_n\to\infty$, and $D_n/S_n\to 0$. Let $c_n=\dfrac{D_n}{S_n(\log S_n)^{1+\beta}}$, $\beta>0$. Then by Theorem Ia, $\sum_n c_n$ is convergent, and $c_n\leqq d_n$. On the other hand, since $D_n\leqq 1$, we have for $\alpha>0$

$$\left[\frac{D_n}{S_n(\log S_n)^{1+\beta}}\right]^{1-\alpha}\geqq\frac{D_n}{S_n^{1-\alpha}[\log S_n]^{(1-\alpha)(1+\beta)}}$$

which by comparison with $D_n/S^{1-\gamma}$, with $0<\gamma<\alpha$, turns out to be divergent.

We have the obvious corollary:

If $d_n>0$, is any sequence such that $\sum_n d_n^e$ is divergent and $0<f<e$, then there exists a sequence $0<c_n\leqq d_n$ such that $\sum_n c_n^f$ is convergent but $\sum_n c_n^g$ is divergent for $g<f$.*

Obviously the Abel-Dini theorem gives:

If $d_n>0$, is any sequence such that $\sum_n d_n^e$ is divergent and $0<f<e$, then there exists a sequence $d_n'\leqq d_n$ such that $\sum_n d_n'^g$ is divergent for $g\leqq f$, but $\sum_n d_n'^g$ is convergent for $g>f$.

* This and the succeeding corollary are due to H. H. Goldstine, who obtained them in proving a theorem on classes of sequences due to E. H. Moore.

ON SEMI-CONTINUITY*

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1. The concept. Semi-continuity is one of many fundamental conceptions that were introduced into Analysis around the year 1900. The following incomplete survey will be concerned primarily with applications of the idea of semi-continuity. The extent of the literature makes the inclusion of a comprehensive bibliography prohibitive, and for our modest purposes a detailed bibliography is irrelevant anyway. Our objective is merely to illustrate how semi-continuity, originally a mere abstraction, became in the course of time an indispensable tool for many classical problems in Analysis, Geometry, and Potential Theory. We restrict ourselves therefore to two remarks concerning literature. The well-known standard treatises on Real Variables and Point-set Theory (Carathéodory, Hahn, Hobson, R. L. Moore) contain comprehensive information on semi-continuity itself, while further details and references concerning applications may be found in two surveys by the present author (*On the problem of Plateau* and *On subharmonic functions*, both in the Ergebnisse der Mathematik series).

For brevity, we shall write l.s.c. and u.s.c. for lower semi-continuous and upper semi-continuous respectively. Let $f(x)$ be a function defined in a closed interval

$$I: a \leqq x \leqq b.$$

Then $f(x)$ is l.s.c. at a point x_0 of I if

$$f(x_0) > -\infty \quad \text{and} \quad f(x_0) \leqq \underline{\lim}\, f(x_n)$$

for every sequence $x_n \to x_0$. Similarly, $f(x)$ is u.s.c. at a point x_0 of I if

$$f(x_0) < +\infty \quad \text{and} \quad f(x_0) \geqq \overline{\lim}\, f(x_n)$$

for every sequence $x_n \to x_0$. If $f(x)$ is l.s.c. (u.s.c.) at every point of I, then $f(x)$ is l.s.c. (u.s.c.) in I.

The conditions involving the values $\pm\infty$ are needed to cover certain applications. More explicitly, we require that $-\infty < f(x) \leqq +\infty$ if $f(x)$ is l.s.c. and $-\infty \leqq f(x) < +\infty$ if $f(x)$ is u.s.c.

For the sake of immediate illustration, the following simple remark may be used. Let $g(x)$ be continuous at a point x_0 of I. Let us take a number $k \neq g(x_0)$. Define

$$f(x) = g(x), \qquad x \neq x_0,$$
$$f(x_0) = k.$$

Clearly, $f(x)$ is discontinuous at x_0. On the other hand, $f(x)$ is semi-continuous there. Specifically, $f(x)$ is l.s.c. at x_0 if $k < g(x_0)$, and u.s.c. at x_0 if $k > g(x_0)$.

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* Presented, by invitation, at the annual meeting of the Mathematical Association of America, in Bethlehem, Pa., January 1, 1942.

If a function $f(x)$ is both l.s.c. and u.s.c. at x_0, then it is continuous at x_0 and conversely. Thus the property of continuity may be thought of as the logical sum of the properties of lower and upper semi-continuity. We expect therefore that semi-continuous functions will enjoy about half of the properties of continuous functions. Several simple and important theorems bear out this expectation. Two typical theorems of this type will be stated now. The simple proofs are omitted. Such theorems may possibly be used to good advantage as exercises in familiarizing beginners with the idea of continuity itself.

If $f(x)$ is l.s.c. (u.s.c.) in I, then $f(x)$ is bounded from below (above) in I.

If $f(x)$ is l.s.c. (u.s.c.) in I, then it takes on there a minimum (maximum).

The class of semi-continuous functions is of course much more comprehensive than the class of continuous functions. Still, in a sense to be explained now, semi-continuous functions are quite close to continuous functions. Let us use the term L-sequence (U-sequence) to refer to a monotonically increasing (decreasing) sequence of functions. We have then the following theorem.

If $f_n(x)$ is an L-sequence (U-sequence) of continuous functions in I, then $\lim f_n(x) = f(x)$ is l.s.c. (u.s.c.) in I. Conversely, if $f(x)$ is l.s.c. (u.s.c.) in I, then we have an L-sequence (U-sequence) of continuous functions $f_n(x)$ such that $f(x) = \lim f_n(x)$ in I.

Thus semi-continuous functions are simply the limits of monotonic sequences of continuous functions. It is then natural to consider the limit functions of monotonic sequences of semi-continuous functions. It turns out that these limit functions comprise all the functions that seem to be needed in applications, namely all measurable functions.* Thus semi-continuous functions are, in a sense, stepping stones between continuous and measurable functions, As a consequence semi-continuous functions may be used to construct a most beautiful and pedagogically sound theory of measurable functions and their integrals.

The conception of semi-continuity can be extended, in an obvious manner, to functions of several variables and more generally to functionals. We state presently the general definition; examples will follow in a moment.

Let $\eta = f(\xi)$ be a functional, defined on a range X, with values in a range Y, Then $f(\xi)$ is l.s.c. (u.s.c.) at ξ_0 in X if

$$f(\xi_0) \leqq \underline{\lim}\, f(\xi_n) \quad \text{for} \quad \xi_n \to \xi_0,$$

or

$$f(\xi_0) \geqq \overline{\lim}\, f(\xi_n) \quad \text{for} \quad \xi_n \to \xi_0$$

respectively. This definition presupposes first that in both ranges X, Y the conception of convergent sequence and that of limit inferior (limit superior) is suitably determined, and second that in the range Y an order relation is given.

We shall discuss now a few outstanding examples of semi-continuous functions and functionals.

* The precise theorem reads as follows. Every measurable function is the limit, almost everywhere, of a U-sequence of l.s.c. functions and of an L-sequence of u.s.c. functions.

2. Potentials and convex functions. We restrict ourselves, for definiteness, to the two-dimensional case. In the xy-plane, let there be given a negative mass-distribution (the reader may think of a distribution of negative electric charges). Such a distribution gives rise to a potential $u(x, y)$. What are the characteristic mathematical properties of this potential? This question is of course of great interest. By virtue of a most important coincidence, the answer to this question is identical to the answer to the following, entirely different, question that originates in pure mathematics. Let $f(x)$ be a function that is convex from below in an interval I. Then, in every sub-interval of I, the graph of $f(x)$ is below its chord. Since the chord is given by a linear function $h(x)=\alpha x+\beta$, a convex function $f(x)$ may also be described as a *sublinear function* in the sense that, in every sub-interval, $f(x)$ is dominated by the linear function $h(x)$ with the same boundary values. In view of the importance of convex functions of a single variable, we may ask for a useful generalization of this idea to functions of two (or several) variables. Now, a linear function $h(x)=\alpha x+\beta$ may also be described as a solution of the differential equation

$$\frac{d^2h}{dx^2} = 0.$$

As a natural generalization, we have then, in the two-dimensional case, the *harmonic functions* which arise as solutions of the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Thus we arrive at the definition of *subharmonic functions*, defined by the property that in every sub-region of its range of definition, a subharmonic function is dominated by the harmonic function with the same boundary values. Now, how is all this related to semi-continuity? First of all, we find that the potentials $u(x, y)$, referred to above, are subharmonic functions. As far as applications in pure mathematics are concerned, we may include in the definition of a subharmonic function any continuity requirements to suit ourselves. However, if all the potentials $u(x, y)$ of negative mass-distributions are to be subharmonic functions and conversely, if every subharmonic function is to be, essentially, a potential of this type, then we have no choice in this matter. It turns out that we have to require precisely the property of upper semi-continuity.

3. Arc-length and area. In the preceding example, semi-continuity was in a sense forced upon the Analyst by an outside application. Similar situations arise if we undertake to apply Analysis to certain simple and fundamental problems in Geometry. Let X designate the class of continuous curves in three-space, say. Let ξ designate an individual curve and $f(\xi)$ the arc-length of ξ, where $f(\xi)$ may be $+\infty$. Simple examples show that $f(\xi)$ is not a *continuous* functional. That is, $\xi_n \to \xi_0$ generally does not imply that $f(\xi_n) \to f(\xi_0)$. But, if convergent sequences of curves are suitably defined, then it is found that $f(\xi)$ is a l.s.c. functional.

Similar statements hold if X stands for the class of continuous surfaces and $f(\xi)$ for the area of the surface ξ. In fact, this semi-continuity property is used as a desideratum in defining the area of a surface. Unfortunately, the area of a surface cannot be defined, in analogy with the arc-length, as the limit of the areas of inscribed polyhedra, as simple examples show. The generally accepted definition of the area $f(\xi)$ of a surface ξ is given by the formula.

$$f(\xi) = \text{gr.l.b.}\ \underline{\lim}\ \phi(\pi_n), \tag{1}$$

where the greatest lower bound is taken with respect to all sequences of polyhedra π_n, inscribed or not, converging to ξ in an appropriate sense, and $\phi(\pi_n)$ designates the elementary area of π_n. The purpose and effect of this definition is to make the area a l.s.c. functional.

We wish to call attention on this occasion to a simple, natural and apparently quite difficult problem. As regards the definition of the area by formula (1), it may be argued that it would be more natural to state that definition in terms of *inscribed* polyhedra. Let us designate by $f^*(\xi)$ the quantity obtained by restricting, in the formula (1), the polyhedra π_n by the requirement that each π_n be inscribed in the surface ξ. The problem we wish to call attention to requires to determine whether $f(\xi)=f^*(\xi)$. Only partial results are known, and in every instance the contrast between the simplicity of the problem and the complexity of the methods is striking. Still, a little reflection shows that the problem would be solved if we could prove that the functional $f^*(\xi)$ is also l.s.c.

4. Calculus of Variations. The analytic formulation of the geometrical facts stated in sections 2 and 3 leads to important questions in Analysis. Let us consider, for definiteness, a sequence of smooth curves given by equations

$$y = g_n(x), \qquad 0 \leqq x \leqq 1.$$

Suppose that $g_n(x) \to g(x)$ uniformly in this interval, where $g(x)$ is again a smooth function. The arc-lengths of all these curves are then given by the familiar integral formula, and the remarks in section 3 lead to the relation

$$\int_0^1 [1 + g'(x)^2]^{1/2} dx \leqq \underline{\lim} \int_0^1 [1 + g_n'(x)^2]^{1/2} dx. \tag{2}$$

Let us observe that our assumptions do not imply any degree of approximation in terms of derivatives. Hence there is nothing obvious about (2) from the point of view of Analysis. We are thus led to the following general question. Let a curve ξ be given by equations of the form

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad 0 \leqq t \leqq 1.$$

Let $F(x, y, z, \lambda, \mu, \nu)$ be a function of six variables. We define a functional

$$f(\xi) = \int_0^1 F(x(t), y(t), z(t), x'(t), y'(t), z'(t))dt.$$

Under what conditions will this functional $f(\xi)$ be l.s.c.? The same question arises in connection with double integrals depending upon a surface ξ. The questions thus raised are important in Calculus of Variations, where we deal with the problem of minimizing such integrals. We mentioned above that a l.s.c. function, defined on a closed interval, reaches a minimum there. We can expect then to obtain existence theorems in Calculus of Variations in cases when the variational integral is a l.s.c. functional. This simple idea is the foundation of the so-called *direct method* in Calculus of Variations. In this manner, semi-continuity takes its place amongst the fundamental conceptions of a classical field in Analysis.

5. Point sets. As a last example, let us consider the following situation. Let X designate a point-set, in the xy-plane for definiteness. Let X be partitioned, in some definitely given manner, into mutually exclusive subsets for which we shall use the generic notation η. We denote the class of all the sets η by Y. We define now a functional $\eta = f(\xi)$, for points ξ in X, by the agreement that $\eta = f(\xi)$ is the unique set η of the class Y that contains the point ξ. Under what conditions shall we say that this functional $f(\xi)$ is u.s.c.? Checking back, we see that we have to state first various definitions. Since ξ stands for a point in the xy-plane, the relation $\xi_n \to \xi_0$ will stand for ordinary convergence. Next, we define $\overline{\lim}\ \eta_n$ as the set of points (x, y) with the property that every neighborhood of (x, y) is entered by infinitely many sets of the sequence η_n. Finally, if S_1, S_2 are two point-sets, we shall interpret the relation $S_1 \leqq S_2$ as meaning that S_1 is a subset of S_2, and we shall write $\subset$ instead of $\leqq$. Applying the general definition stated in section 1, we shall say that the functional $\eta = f(\xi)$, defined at the beginning of the present section, is u.s.c. if for every ξ_0 in X

$$f(\xi_0) \supset \overline{\lim}\ f(\xi_n),$$

whenever $\xi_n \to \xi_0$. The prevalent practice, though, is to describe the class Y of subsets η as an *upper semi-continuous collection*. The wording of the definition is of course irrelevant so long as the logical content is the same. The point is that semi-continuity, applied in this manner in point-set theory, admits of many significant applications.

The preceding rapid survey lends support to the statement that a mathe matical conception may originate as a mere abstraction, and still develop into a tool of great usefulness in a surprising variety of applications. To avoid giving undue aid and comfort to the specialist dealing in mere abstractions, we want to conclude by a complementary statement: it would seem that only those mere abstractions survive that do develop into tools of great usefulness in a surprising variety of applications.

THE RATIONAL CANONICAL FORM OF A MATRIX*

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1. Introduction. There are several different methods in the literature for the computation of the rational canonical form of a matrix.† We shall give a method here which is (1) valid in an arbitrary field K, (2) exceedingly simple, and (3) based on elementary transformations. Our result is obtained by exploiting the same lemma which Albert has used in simplifying the computation of the inverse of a non-singular matrix.‡ A description of the general reduction process is given in Section 2 and a particular numerical example is treated in Section 3.

2. The reduction process. Two matrices A and B with elements in K are said to be similar (in K) in case $B = PAP^{-1}$ with P a non-singular matrix with elements in K. Now a non-singular matrix is the product of a finite number of elementary transformation matrices,§ and the lemma used by Albert then shows that A is similar to B if and only if we can obtain B from A by a finite number of *elementary similarity transformations* of the following types.

Type (i). *Interchanging the ith and jth rows and the corresponding columns.*

Type (ii). *Adding $c \epsilon K$ times the ith row to the jth row and subtracting c times the jth column from the ith column.*

Type (iii). *Multiplying the ith row by a nonzero element $a \epsilon K$ and multiplying the ith column by a^{-1}.*

We shall denote these transformations by e_{ij}, $p_{ij}(c)$, and $r_i(a)$, respectively.

Let us call a matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_2 & a_1 \end{pmatrix} \tag{1}$$

a *canonical block*. We shall show how a matrix $A = (a_{ij})(i, j = 1, \cdots, n)$ may be reduced by transformations of types (i)–(iii) to a matrix $B = \text{diag}\{B^{(1)}, \cdots, B^{(t)}\}$ where $B^{(\alpha)}(\alpha = 1, \cdots, t)$ are canonical blocks. The computation of the rational canonical form of B may then be carried through via elementary similarity trans-

* Presented to the American Mathematical Society, Dec. 29, 1941, under the title Elementary similarity transformations and the rational canonical form of a matrix.

† C. C. MacDuffee lists the contributions prior to 1933 in The Theory of Matrices, Ergebnisse der Mathematik, vol. 2, 1933, p. 424. See also E. T. Browne, On the reduction of a matrix to canonical form, this MONTHLY, vol. 47, 1940, pp. 437–450.

‡ A rule for computing the inverse of a matrix, this MONTHLY, vol. 48 1941, pp. 198–199. See also A. A. Albert, Introduction to Algebraic Theories, Chicago, 1941, p. 42; and G. Birkhoff and S. Mac Lane, A Survey of Modern Algebra, New York, 1941, p. 279.

§ A. A. Albert, Introduction to Algebraic Theories, p. 44.

formations. However, it is much simpler to resort to a computation of the invariant factors of $xI-B$. Let $f_\alpha(x)(\alpha=1, \cdots, t)$ be the non-trivial invariant factor of $xI-B^{(\alpha)}$. Let $d(x)$ be the greatest common divisor of $f_1(x), \cdots, f_t(x)$. Let $B_1^{(t)}, B_1^{(t-1)}$ be canonical blocks corresponding to $d(x)$ and to $f_t(x)\cdot f_{t-1}(x)/d(x)$ (See Albert, *Introduction to Algebraic Theories*, p. 105). Then $B_1=\operatorname{diag}\{B^{(1)}, \cdots, B^{(t-2)}, B_1^{(t-1)}, B_1^{(t)}\}$ is similar to B. Repeat this process on the matrix $\operatorname{diag}\{B^{(1)}, \cdots, B^{(t-2)}, B_1^{(t-1)}\}$ and continue repetitions until the rational canonical form of B is obtained.

Let us return, then, to the problem of reducing $A=(a_{ij})(i, j=1, \cdots, n)$ to $\operatorname{diag}\{B^{(1)}, \cdots, B^{(t)}\}$ with each $B^{(\alpha)}(\alpha=1, \cdots, t)$ a canonical block. First, if some $a_{1j}(j>1)$ is nonzero, we may apply e_{2j} to obtain a matrix (a'_{ij}) with $a'_{12}\neq 0$. Application of $r_2(a'_{12})$, followed by $p_{12}(a'_{11}), p_{32}(a'_{13}), \cdots, p_{n2}(a'_{1n})$ yields a matrix (a''_{ij}) whose first row is $(0, 1, 0, \cdots, 0)$. If then some $a''_{2k}(k>2)$ is nonzero, we may apply e_{3k} to obtain a matrix (a'''_{ij}) with $a'''_{23}\neq 0$. Application of $r_3(a'''_{23})$ followed by $p_{13}(a'''_{21}), p_{23}(a'''_{22}), p_{43}(a'''_{24}), \cdots, p_{n3}(a'''_{2n})$ produces a matrix whose first two rows agree with (1). This process may be repeated until we have reduced the matrix A to its rational canonical form of just one canonical block, in which case we have attained our goal; or until we reach a matrix of the form

$$\begin{pmatrix} B_1 & 0 \\ C & D \end{pmatrix}$$

in which B_1 is a $q\times q$ canonical block. Using transformations of type (ii), we can replace this matrix by one of the same form in which the last $q-1$ columns of the matrix corresponding to $C=(c_{rs};\ r=1, \cdots, n-q,\ s=1, \cdots, q)$ are all zero. How this is done will be clear if we show how to "remove" the element c_{1q}. To do this, multiply the $(q-1)$st row by c_{1q} and subtract it from the $(q+1)$st row and add c_{1q} times the $(q+1)$st column to the $(q-1)$st column. Note that this transformation is $p_{q-1,q+1}(-c_{1q})$ and that it leaves the matrix B_1 unaltered. When the last $q-1$ columns of the matrix C have been replaced by zeros, it may be that each element in the first column of the matrix corresponding to C is zero, and then we have reduced A to the matrix $\operatorname{diag}\{B^{(1)}, D^{(1)}\}$, with $B^{(1)}=B_1$ and $D^{(1)}$ the matrix corresponding to D. If so, we apply the above process to the matrix $D^{(1)}$. If not, some element c'_{1s} of the matrix $C^{(1)}=(c'_{rs})$ corresponding to C is nonzero. Apply $r_{q+s}(1/c'_{1s})$, followed by $p_{q+s,q+1}(-c'_{11}), \cdots, p_{q+s,q+s-1}(-c'_{1,s-1}), p_{q+s,q+s+1}(-c'_{1,s+1}), \cdots, p_{q+s,n}(-c'_{1,n-q}), e_{q+1,q+s}$ to obtain a matrix

$$\begin{pmatrix} B_1 & 0 \\ C^{(2)} & D^{(2)} \end{pmatrix}$$

in which the only nonzero element of $C^{(2)}=(c''_{rs})$ is $c''_{11}=1$. In this case it is always possible to obtain a matrix with a larger canonical block in the upper left-hand corner. The argument falls into two cases according to whether or not there is some nonzero element $d''_s\ (s>1)$ of the matrix $D^{(2)}=(d''_{us};\ u,\ s=1, \cdots, n-q)$. Since the argument is similar in the two cases, we shall give the

details of proof only in the more involved case in which there is some nonzero element $d''_{1s}(s>1)$. Apply $e_{q+2,q+s}$ $r_{q+2}(d''_{1s})$, followed by $p_{q+3,q+2}(d''_{13}), \cdots, p_{n,q+2}(d''_{1n-q})$ to obtain a matrix

$$\begin{pmatrix} B_1 & 0 \\ C^{(2)} & D^{(3)} \end{pmatrix}$$

in which the first row of $D^{(3)}$ is $(\beta, 1, 0, \cdots, 0)$. Now apply $e_{1,q+1}, e_{2,q+1}, \cdots, e_{q,q+1}$ to obtain the matrix

$$(2) \qquad \begin{pmatrix} \beta & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \cdots 0 \\ 0 & a_q & a_{q-1} & \cdots & a_2 & a_1 & 0 & 0 \cdots 0 \\ * & 0 & 0 & \cdots & 0 & 0 & * & * \cdots * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & 0 & 0 & \cdots & 0 & 0 & * & * \cdots * \end{pmatrix}$$

Here the asterisks designate possibly nonzero elements. Applying $p_{q+2,2}(1)$ introduces new elements into the second row; but it is easy to remove these elements, new elements being introduced in the third row. Then these elements and the subsequent elements may be similarly removed without affecting the element 1 in the nth row and the $(n+1)$st column. The same process may be employed to remove β. We are thus assured a larger canonical block in these cases. Since the matrix A has only a finite number of rows and columns, our process must ultimately cease and we have reduced the matrix A to the desired form.

3. An example. As an illustration of our reduction process of Section 2 we shall reduce the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{pmatrix}$$

to canonical form. Note that A is singular and that it has the form (for $q=1$) which led to (2). We apply e_{12}, $p_{32}(1)$, and $p_{12}(2)$ to obtain successively

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & -1 & 3 & -1 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 3 & -1 \\ 3 & 0 & 2 & -1 \\ 5 & 0 & 3 & -1 \end{pmatrix}.$$

Application of $r_3(3)$, $p_{43}(-1)$, and $p_{13}(5)$ gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 1 & -1 \\ 9 & 0 & 2 & -3 \\ 5 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 4 & 0 & 1 & -1 \\ 5 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 5 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Use of $p_{23}(1)$, $r_4(-1)$, and $p_{14}(-1)$ yields

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 4 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 4 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Finally, we use $p_{24}(4)$ and $p_{34}(2)$ to find

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

The last matrix obtained is the rational canonical form of A.

A MAXIMUM PROBLEM

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If an impurity appears at a point P in a solid, the question may be asked: What plane section cutting off P will leave a maximum volume of pure material? For example, the problem would seem to have a certain practical aspect for the gem-cutter. There is a simple geometric way to characterize the answer.

Given a set M of a finite number of bounded solids, fixed in position in three-space and a point P in the space; then a plane p through P divides M into two subsets on the volume of one of which attention is fixed by the label V_p. If this volume is the same as that of M, the plane p determines an absolute maximum for the function V. Otherwise consider the two cases I and II according as P is not, or is, in the same plane as a plane face of one of the solids of M. Let the plane p have in common with the set M the area A_p. Using these terms it is possible to state the following theorem:

THEOREM. *In case I a necessary condition that the plane p serve to maximize the function V is that P be the centroid of the area A_p.*

For the proof let the plane p be fixed; let L be a variable line in p through P; let q be a plane intersecting p in L and making an angle θ with the plane p;

then the volume V_q corresponding to V_p may be described as $V(\theta, L)$. Let the line L be fixed. Let L divide the area A_p into areas A_{p1} and A_{p2} and divide the corresponding area A_q into areas A_{q1} and A_{q2} where A_{p1} and A_{q1} are to be separated by an acute angle θ. Let $\bar{x}_{p1}$, $-\bar{x}_{p2}$, $\bar{x}_{q1}$, $-\bar{x}_{q2}$ be the distances of the centroids of A_{p1}, A_{p2}, A_{q1}, A_{q2}, respectively, from the line L. Denote the portion of M between A_{p1} and A_{q1} by $V_1(\theta, L)$ and the portion between A_{p2} and A_{q2} by $V_2(\theta, L)$.

Let us investigate the existence of the derivatives $\partial V_1/\partial\theta$ and $\partial V_2/\partial\theta$ for $\theta=0$ and any fixed L. Let the angle θ be taken small, say, less than $\Delta\theta$. Since we are considering case I, $A_{q1}(\theta)$ is a continuous function, and therefore $\bar{x}_{q1}(\theta)$ and $A_{q1}(\theta)\bar{x}_{q1}(\theta)$ are continuous. Thus if s and S indicate the minimum and maximum values of $A_{q1}(\theta)\bar{x}_{q1}(\theta)$ in the interval $-\Delta\theta\leqq\theta\leqq\Delta\theta$, we know that s and S may be made to differ from $A_{p1}\bar{x}_{p1}$ by an arbitrarily small number simply by taking $\Delta\theta$ sufficiently small. Consider the inequality:

$$s\Delta\theta \leqq \Delta V_1 \leqq S\Delta\theta$$

where the bounding approximations are the volumes of solids of revolution expressed by the Theorem of Pappus. Then divide by $\Delta\theta$ and let $\Delta\theta$ approach 0 to show that $\partial V_1/\partial\theta$ exists at $\theta=0$ and has the value $A_{p1}\bar{x}_{p1}$. In similar fashion $\partial V_2/\partial\theta=-A_{p2}\bar{x}_{p2}$.

Consider the total variation $\Delta V=V_q-V_p=\Delta V_1-\Delta V_2$. Hence

$$\frac{\partial V}{\partial\theta}=\frac{\partial V_1}{\partial\theta}-\frac{\partial V_2}{\partial\theta}=A_{p1}\bar{x}_{p1}+A_{p2}\bar{x}_{p2}.$$

Then since $\partial V/\partial\theta$ exists at $\theta=0$, a necessary condition that p serve to maximize $V(\theta, L)$ is that $\partial V/\partial\theta=0$; but this is to assert that the line L passes through the centroid of the area $A_p=A_{p1}+A_{p2}$. If this argument is pursued for each of the lines L in p and through P, the demand is that P itself be the centroid of the area A_p. This completes the proof.

In case II the above argument fails for then the function $A_{q1}(\theta)$ may have discontinuities with a saltus possible when the variable plane through P is in coincidence with a portion of the surface of M which is flat. Thus $\partial V_1/\partial\theta$ may fail to exist for the critical plane p which maximizes V, and then this plane need bear no nice relation to the point P. But even in case II a maximizing plane p can sometimes be discovered by noticing that for it the left-hand and right-hand derivatives $\partial V/\partial\theta$ exist, are evaluated by $(A_p\bar{x}_p)^-$ and $(A_p\bar{x}_p)^+$, and are respectively positive and negative, creating the "tent" type of maximum.

In general, the function $V(\theta, L)$ is continuous, bounded by the volume of M, and periodic of period 2π in each of the variables. If $\overline{V}$ indicates the complement of V in M, then $V(\theta+\pi, L)=\overline{V}(\theta, L)$, so that V may be described as skew symmetric of period π in the variable θ.

The condition of the principal theorem was described as necessary, but of course is not sufficient, for a maximum of V; for it characterizes equally well

horizontal points of inflection of the volume function. Minimum points can all be found by virtue of the skew symmetric property already mentioned.

We state next the form which the theorem takes for two-space leaving the proof to the interested reader since he can proceed in a manner analogous to that given above, using that variation of the Theorem of Pappus which gives the plane area swept out by one or more segments of a rotating line.

Given the set M of a finite number of bounded areas in a plane; P, a point in the plane; and L, a variable line through P, separating M into two subsets, on the area of one of which attention is fixed by the label A_L. If this area is the same as that of M, then L determines an absolute maximum for the function A. Otherwise two cases I and II arise according as P is not, or is, on a line, a portion of which serves as part of the boundary of M. Let the line L have in common with M the line segments S_L.

THEOREM. *In case I a necessary condition that the line L serve to maximize the function A is that P be the centroid of the line segments* S_L.

If M be a single area which is convex with respect to P (*i.e.*, every line through P intersects the boundary of M in just two points, separated by P), then the necessary condition is that the line L maximizing A_L have the line segment S_L bisected at P. But this is a known result* which can be established in various ways. One of these, which expresses A_L as an integral containing a parameter, namely—the slope of the line through P, offers an interesting exercise in the use of Leibniz' rule for differentiating such integrals. A posteriori, it follows for this particular type of area, granting the existence of a maximum for A_L, that there *exists* a line segment S_L bisected at P. This is a theorem established elsewhere by other means and for far more general boundary curves.†

In conclusion let me suggest a further line of theorems which can be proved by a slight modification of the method of this paper. For example for the set of coplanar areas M let the problem be to determine that tangent line L of a curve C with continuously turning tangent which will separate M into two subsets, making the area of one of these a maximum. A necessary condition (in case I) is that the point of tangency P shall be the centroid of the segments S_L in common to L and M. As a special case there is the well known theorem that a triangle of minimum area circumscribing an oval must have the points of tangency bisecting the sides. The generalization of these statements to three dimensions is straightforward.

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* For example, see H. Levy, Modern Science, Knopf, 1939, p. 317.

† J. D. Hill, H. E. Vaughan, this MONTHLY, vol. 46, 1939, p. 657.

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

NEW FORMS OF CERTAIN INTEGRALS

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It is a remarkable fact, sufficient in itself perhaps to justify such investigations, that often inquiries into relatively advanced or abstract questions throw new light upon topics long regarded as entirely familiar. The present paper is a case in point. In attempting to systematize certain integrations involving Bessel functions the author encountered formulas which proved to have useful and apparently new analogues in the integration of circular and exponential functions. It is the purpose of this note to call attention to these results and to sketch the process by which they may be derived.

We consider first the integral $\int x^n \cos x\, dx$, and guided by a few particular cases we write

$$\int x^n \cos x\, dx = P(x) \sin x + Q(x) \cos x, \tag{1}$$

where P and Q are functions of x to be determined. Differentiation of (1) gives

$$x^n \cos x = P' \sin x + P \cos x + Q' \cos x - Q \sin x. \tag{2}$$

Since the antiderivative of a function is unique to within an additive constant, any pair of functions P, Q satisfying (2) can be used in (1). A sufficient condition that P and Q satisfy (2) is that*

$$P' - Q = 0, \tag{3}$$

$$P + Q' = x^n. \tag{4}$$

Hence we seek a solution of these equations. Eliminating Q from (3) and (4) we have

$$P'' + P = x^n. \tag{5}$$

* It is an interesting exercise in calculus to show that the more general equations,

$$P' - Q = F(x) \cos x,\ P + Q' = x^n - F(x) \sin x,$$

where $F(x)$ is arbitrary, afford a solution of (1) differing only by a constant from that given by (3) and (4).

We solve this by assuming

$$P = x^c(a_0 + a_1x + a_2x^2 + \cdots).$$

It turns out that we must choose $c = n+2$. Straightforward calculations then yield at once

$$P \equiv x^{n+2}\left\{\frac{1}{(n+1)(n+2)} - \frac{x^2}{(n+1)\cdots(n+4)} + \cdots\right\},$$

or, after multiplying up and down by $\Gamma(n+1)$,

$$P = \Gamma(n+1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k}}{\Gamma(n+2k+1)}, \tag{6}$$

and from (3)

$$Q = \Gamma(n+1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k-1}}{\Gamma(n+2k)}. \tag{7}$$

Equation (1) with P and Q defined by (6) and (7) respectively is valid for all values of n except $n = -1, -2, -3, \cdots$, and has the useful property of converging more rapidly than the formula obtained by multiplying the Maclaurin expansion of $\cos x$ by x^n and integrating termwise.

Particularly useful and elegant results are obtained when n is a positive integer. If, for instance, n is even, we have

$$P = n!\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k}}{(n+2k)!} = n!(-1)^{(n+2)/2}\sum_{j=(n+2)/2}^{\infty}(-1)^j\frac{x^{2j}}{(2j)!}$$
$$= (-1)^{(n+2)/2}n!\left[\cos x - S_{(n+2)/2}(\cos x)\right],$$

$$Q = n!\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{n+2k-1}}{(n+2k-1)!} = n!(-1)^{n/2}\sum_{j=n/2}^{\infty}(-1)^j\frac{x^{2j+1}}{(2j+1)!}$$
$$= (-1)^{n/2}n!\left[\sin x - S_{n/2}(\sin x)\right],$$

where $S_r(\cos x)$ denotes the rth partial sum of the Maclaurin series for $\cos x$. Finally, then,

$$\int x^n \cos x\,dx = (-1)^{n/2}n!\left[S_{(n+2)/2}(\cos x)\sin x - S_{n/2}(\sin x)\cos x\right], \tag{8}$$
$$n \text{ even}, n > 0.$$

To this are to be added these companion formulas, all obtained in the same fashion:

$$\int x^n \cos x\,dx = (-1)^{(n-1)/2}n!\left[S_{(n+1)/2}(\sin x)\sin x + S_{(n+1)/2}(\cos x)\cos x\right], \tag{9}$$
$$n \text{ odd}, n > 0,$$

$$\int x^n \sin x\,dx = (-1)^{(n+2)/2} n!\left[S_{n/2}(\sin x)\sin x + S_{(n+2)/2}(\cos x)\cos x\right], \tag{10}$$

n even, $n > 0$,

$$\int x^n \sin x\,dx = (-1)^{(n-1)/2} n!\left[S_{(n+1)/2}(\cos x)\sin x - S_{(n+1)/2}(\sin x)\cos x\right], \tag{11}$$

n odd, $n > 0$,

$$\left.\begin{aligned} \int x^n e^x dx &= (-1)^n n! S_{n+1}(e^{-x})e^x \qquad (12)\\ \int x^n e^{-x} dx &= -n! S_{n+1}(e^x)e^{-x} \qquad (13)\end{aligned}\right\},$$

n integer, $n > 0$.

These seem to the writer to be superior to the usual one-step reduction formulas to be found in most handbooks.

The application to Bessel functions which first suggested the above results to the author involved the integral $\int x^n J_m(x)dx$. If we write

$$\int x^n J_m(x)dx = xP(x)J_m(x) + xQ(x)J_{m-1}(x) \tag{14}$$

and differentiate we have

$$\begin{aligned} x^n J_m = {} & PJ_m + xP'J_m + xP\left(J_{m-1} - \frac{m}{x}J_m\right) + QJ'_{m-1} + xQ'J'_{m-1} \\ & + xQ\left(\frac{m-1}{x}J_{m-1} - J_m\right), \end{aligned} \tag{15}$$

where the expressions in parentheses are well-known equivalents of J'_m and J'_{m-1} respectively. A sufficient condition that (15) subsist is that simultaneously

$$xP' + (1-m)P - xQ = x^n, \tag{16}$$

$$xQ' + mQ + xP = 0. \tag{17}$$

If P and Q are eliminated from these equations we find

$$P'' + \frac{P'}{x} + \left[1 - \frac{(m-1)^2}{x^2}\right]P = (n-m-1)x^{n-2}, \tag{18}$$

$$Q'' + \frac{Q'}{x} + \left[1 - \frac{m^2}{x^2}\right]Q = -x^{n-1}. \tag{19}$$

P and Q are thus to be determined as particular solutions of non-homogeneous Bessel equations of orders $(m-1)$ and m respectively. In general these solutions will be what are known as Lommell functions. We are interested, however, in

the following rather common special cases in which it turns out that P and Q can be expressed in terms of more familiar and accessible functions:

$$\left.\begin{array}{ll} A.1 & n = m + 2k + 1 \\ A.2 & n = -m + 2k + 1 \end{array}\right\} \quad k = 0, 1, 2, \cdots,$$

$$\left.\begin{array}{ll} B.1 & n = m + 2k \\ B.2 & n = -m + 2k \end{array}\right\} \quad k = 0, \pm 1, \pm 2, \cdots.$$

In each of these cases we assume a solution of the form $Q = x^c(a_0 + a_1x + \cdots)$. In $A.1$ we take $c = m + 2k + 2$ and find at once from (19)

$$Q = -x^{m+2k+2}\left\{\frac{1}{2^2(k+1)(m+k+1)} - \frac{x^2}{2^4(k+1)(k+2)(m+k+1)(m+k+2)} + \cdots\right\}.$$

If we multiply up and down by $2^{m+2k}k!\,\Gamma(m+k+1)$ we have

$$Q = (-1)^k 2^{m+2k} k!\,\Gamma(m+k+1)\left[J_m - S_{k+1}(J_m)\right]. \tag{20}$$

Then from (17) we find

$$P = (-1)^{k+1} 2^{m+2k} k!\,\Gamma(m+k+1)\left[J_{m-1} - S_{k+1}(J_{m-1})\right]. \tag{21}$$

Substituting (20) and (21) into (14) and observing that products of the Bessel functions cancel, we obtain the useful result

$$\int x^{m+2k+1} J_m\,dx = (-1)^k 2^{m+2k} k!\,\Gamma(m+k+1)\left[S_{k+1}(J_{m-1})J_m - S_{k+1}(J_m)J_{m-1}\right]x. \tag{22}$$

With proper care (22) can be used for all values of m. The apparently troublesome cases arising when m is a negative integer and $m+k+1 \leqq 0$ are best handled however by writing $J_{-m} = (-1)^m J_m$ in the original integral and considering only $m = 1, 2, 3, \cdots$.

In A.2 we must have $c = -m+2k+2$. P and Q are then

$$Q = (-1)^k 2^{-m+2k} k!\,\Gamma(-m+k+1)\left[J_{-m} - S_{k+1}(J_{-m})\right],$$

$$P = (-1)^k 2^{-m+2k} k!\,\Gamma(-m+k+1)\left[J_{-m+1} - S_k(J_{-m+1})\right].$$

Substituting these values into (14) and observing that products of the Bessel functions combine to give precisely the Wronskian of the solution-pair $[J_m, J_{-m}]$, that is C/x, and hence contribute only to the integration constant we find

$$\int x^{-m+2k+1} J_m\,dx = (-1)^{k+1} 2^{-m+2k} k!\,\Gamma(-m+k+1)x\left[S_k(J_{-m+1})J_m + S_{k+1}(J_{-m})J_{m-1}\right]. \tag{23}$$

With the proper interpretation (23) can be applied without restriction on m. However the apparently troublesome cases when $m=1, 2, 3, \cdots$ and $-m+k+1 \leqq 0$ are best handled by writing $J_m=(-1)^m J_{-m}$ in the original integral and considering only $m=-1, -2, \cdots$.

In B.1 let $c=m+2k+1$. We then find that

$$Q = (-1)^{k+1}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})2^{m+2k-1}[H_m \mp S_k(H_m)],$$

$$P = (-1)^{k}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})2^{m+2k-1}[H_{m-1} \mp S_k(H_{m-1})],$$

and finally

$$\int x^{m+2k}J_m dx = (-1)^{k+1}\Gamma(k+\tfrac{1}{2})\Gamma(m+k+\tfrac{1}{2})x2^{m+2k-1} \cdot [-J_m\{H_{m-1} \mp S_k(H_{m-1})\} + J_{m-1}\{H_m \mp S_k(H_m)\}], \tag{24}$$

where H_m is Struve's function of order m,* and the plus or minus sign is to be used according as the subscript of S_k is minus or plus.† This result is valid for all m except when m is half an odd integer and simultaneously $m+k+\frac{1}{2} \leqq 0$. In these instances a logarithmic term is inevitable.

In B.2 we take $c=m+2k+1$ and obtain finally

$$\int x^{-m+2k}J_m dx = (-1)^{k+1}2^{-m+2k-1}\Gamma(k+\tfrac{1}{2})\Gamma(-m+k+\tfrac{1}{2})x[J_m\{H_{-m+1} \mp S_{k-1}(H_{-m+1})\} + J_{-m+1}\{H_{-m} \mp S_k(H_{-m})\}], \tag{25}$$

where again the plus or minus sign is to be used according as the subscript of S_k is minus or plus. This result is valid for all m although inconvenient to use when m is half an odd integer and $-m+k+\frac{1}{2} \leqq 0$ because of the indeterminate forms which occur.

The present analysis may of course be used to obtain formulas expressing the integral in (14) in terms of J_m and J_{m+1}. In cases A.1 and A.2 these are perhaps of sufficient interest to be included here.

$$\int x^{m+2k+1}J_m dx = (-1)^k 2^{m+2k}k!\Gamma(m+k+1)[S_{k+1}(J_m)J_{m+1} - S_k(J_{m+1})J_m]x. \tag{22.1}$$

$$\int x^{-m+2k+1}J_m dx = (-1)^k 2^{-m+2k}k!\Gamma(-m+k+1)[S_{k+1}(J_{-m})J_{m+1} + S_{k+1}(J_{-m-1})J_m]x. \tag{23.1}$$

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* Watson, H. N., Theory of Bessel Functions, p. 328.

† By the symbol $S_{-k}(\sum_0^\infty a_i x^i)$ we mean $\sum_{-1}^{-k} a_i x^i$.

NOTE ON WHITTAKER'S METHOD FOR THE ROOTS OF A POWER SERIES

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Several years ago, in running through back issues of the *Proceedings of the Edinburgh Mathematical Society*, the writer found in a paper of the English mathematician E. T. Whittaker a certain, very elegant theorem that deserves more attention than it has evidently received.*

The root of the equation

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{1}$$

which is the smallest in absolute value is given by

$$x = -\frac{a_0}{a_1} - \frac{a_0^2 a_2}{a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}$$

$$- \frac{a_0^4 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \cdots . \tag{2}$$

This theorem is of interest both for the neat manner in which the root is displayed as a function of the coefficients of the equation and as a means of determining the roots of the algebraic and transcendental equations encountered in the solution of technical problems. In this latter respect, the writer has found use of it usually preferable to the other methods available: for no differentiations, cumbersome divisions, or tables of transcendental functions are required; by depressing the roots by one synthetic division each root in turn can be made smallest and by a slide-rule or calculating machine be computed rapidly to the desired degree of exactness. As, often, in the preliminary design of electrical networks or coupled mechanical systems, or in the analysis of their subsequent operation, numerous formidable algebraic or transcendental equations must be solved, the time saved by use of a calculating machine may be considerable. If a ten or twenty inch slide-rule provides sufficient accuracy, the saving is yet greater.

*E. T. Whittaker, A formula for the solution of algebraic or transcendental equations, Proceedings of the Edinburgh Mathematical Society, vol. 36, 1917–18, pp. 103–106.

The following examples serve to indicate the scope and the flexibility of (2).

I. In the calculation of the transient field current of a synchronous machine by Heaviside's expansion theorem,* it was necessary to find the roots of

$$p^5 + 0.6193\,p^4 + 0.1273\,p^3 + 1.044 \times 10^{-2}\,p^2 + 2.966 \times 10^{-4}\,p + 5.739 \times 10^{-7} = 0.$$

By (2) the root smallest in absolute value is

$$p = -\frac{5.739 \times 10^{-7}}{2.966 \times 10^{-4}} - \frac{(5.739 \times 10^{-7})^2\, 1.044 \times 10^{-2}}{2.966 \times 10^{-4} \begin{vmatrix} 2.966 \times 10^{-4} & 1.044 \times 10^{-2} \\ 5.739 \times 10^{-7} & 2.966 \times 10^{-4} \end{vmatrix}} - \cdots$$

$$= -1.936 \times 10^{-3} - 0.1415 \times 10^{-3} = -0.002078.$$

The indicated computation, carried out on a ten inch slide-rule in less than three minutes, yields a value sufficiently accurate for engineering purposes. The known, more exact value (-0.002084) could be obtained by computing a third term.

II. Determine the least root of

$$J_0(x) = 1 - x^2/2^2 + x^4/2^2 \cdot 4^2 - x^6/2^2 \cdot 4^2 \cdot 6^2 + \cdots$$
$$= 1 - z/2^2 + z^2/2^2 \cdot 4^2 - z^3/2^2 \cdot 4^2 \cdot 6^2 + \cdots .$$

As the form of the coefficients enables one to manipulate the determinants with ease, (2) quickly yields

$$z = 4 + 4/3 + 20/57 + 316/4009 = 5.763;\ x = 2.40+.$$

The value given in standard tables is 2.405.

Note by the Editor. Whittaker gives a proof of his theorem only for the case in which $f(x)$ is a polynomial. The combined efforts of R. P. Agnew, Barkley Rosser, and myself have sufficed to prove the general theorem in the following more useful, though less spectacular, form.

THEOREM. *Let $f(z)$ be an analytic function, and r a complex number, such that*

(i) $f(0) = -1$,

(ii) *r is a simple root of $f(z) = 0$,*

(iii) *If s is a root of $f(z) = 0$ distinct from r then $|s| > |r|$,*

(iv) *$f(z)$ has no singular points, except possibly poles, in or on the circle $|z| = |r|$.*

Let $-1 + a_1 z + a_2 z^2 + \cdots$ be the Maclaurin expansion for $f(z)$, and define $A_0, A_1, \cdots$ by

$$A_0 = 1, \quad A_n = a_1 A_{n-1} + a_2 A_{n-2} + \cdots + a_n A_0, \qquad n > 0. \tag{3}$$

Then $r = \lim_{n\to\infty} A_n / A_{n+1}$.

* Doherty and Keller, Mathematics of Modern Engineering, p. 129.

Proof. We first prove a purely formal identity. Namely, if α is any complex number and if $g(z)=(1-z/\alpha)/f(z)=b_0+b_1z+b_2z^2+\cdots$, then $A_n=\alpha^{-n}(b_0+b_1\alpha+b_2\alpha^2+\cdots+b_n\alpha^n)$. Define $F(z)$ by $F(z)=A_0+A_1z+A_2z^2+\cdots$. Then from (3) we have $f(z)F(z)=-1$. Hence

$$F(z) = -1/f(z) = g(z)/(1 - z/\alpha),$$

that is

$$A_0 + A_1z + A_2z^2 + \cdots = (b_0 + b_1z + b_2z^2 + \cdots)(1 + z/\alpha + z^2/\alpha^2 + \cdots)$$
$$= b_0 + (b_0/\alpha + b_1)z+(b_0/\alpha^2 + b_1/\alpha + b_2)z^2 + \cdots .$$

The desired expression for A_n follows on equating corresponding coefficients. (The above relations between power series merely express finite algebraic relations between the coefficients, and so questions of convergence need not be considered.)

Now take $\alpha=r$. By conditions (ii), (iii), and (iv) there is a circle $|z|=R$ inside of which $f(z)$ has the one and only one root r and no singularities except poles. It follows that $g(z)=-(1-z/r)/f(z)$ is regular for $|z|<R$ and so $g(r)$ exists, is not zero, and equals $b_0+b_1r+b_2r^2+\cdots$. We have therefore

$$A_n/A_{n+1} = r\,\frac{b_0 + b_1r + \cdots + b_nr^n}{b_0 + b_1r + \cdots + b_{n+1}r^{n+1}},$$

so that

$$\lim_{n\to\infty} A_n/A_{n+1} = r\,\frac{g(r)}{g(r)} = r.$$

This statement of the theorem is convenient for calculational purposes. That it is equivalent to Whittaker's formulation follows from the fact that for $a_0=-1$ we have

$$A_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_1 \end{vmatrix},$$

and A_n/A_{n+1} is the sum of the first $n+1$ terms of (2).

The following examples indicate how much easier it is to use the new form of the theorem than the old.

III. To solve $\log(1+z)=1/2$ we find the least root of

$$-1 + 2z - \tfrac{2}{2}z^2 + \tfrac{2}{3}z^3 - \tfrac{2}{4}z^4 + \cdots = 0.$$

Thus

$$A_n = 2(A_{n-1} - \tfrac{1}{2}A_{n-2} + \tfrac{1}{3}A_{n-3} + \cdots).$$

We find successively

$$A_0 = 1,\ A_1 = 2,\ A_2 = 3,\ A_3 = 14/3,\ A_4 = 43/6,\ A_5 = 166/15,\ A_6 = 767/45,$$

and so

$$A_0/A_1 = .5,\ A_1/A_2 = .667,\ A_2/A_3 = .643,\ A_3/A_4 = .651,\ A_4/A_5 = .648,\ A_5/A_6 = .6493.$$

The root is $e^{1/2} - 1 = .64872$.

IV. To find the least root of $J_0(\sqrt{z}) = 0$ (see example II) it is convenient to put $B_n = 4^n(n!)^2 A_n$ so as to avoid fractions. Then

$$B_n = \binom{n}{1}^2 B_{n-1} - \binom{n}{2}^2 B_{n-2} + \cdots,$$

$$A_n/A_{n+1} = 4(n+1)^2 B_n/B_{n+1},$$

and we obtain for $B_0, B_1, \cdots, B_7$, the values 1, 1, 3, 19, 211, 3651, 90921, 3081513. These give for $A_0/A_1, \cdots, A_6/A_7$ the values 1, 5.333, 5.684, 5.763, 5.779, 5.782, 5.783. The root of $J_0(x)$ is then $\sqrt{5.783} = 2.405-$.

V. The function

$$-\frac{\tan\sqrt{z}}{\sqrt{z}} = -1 - \frac{1}{3}z - \frac{2}{15}z^2 - \frac{17}{315}z^3 - \frac{62}{2835}z^4 - \frac{1382}{155925}z^5 - \frac{21844}{6081075}z^6 + \cdots$$

has $\pi^2 = 9.870$ as its smallest root. We obtain for $A_0, A_1, \cdots, A_6$, the values 1, $-1/3$, $-1/45$, $-2/945$, $-1/4725$, $-2/93555$, $-1382/648750375$; and for $A_0/A_1, \cdots, A_5/A_6$, the values -3, 15, 10.5, 10, 9.9, 9.877.

R. J. W.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

WAR TOPICS IN CLUB REPORTS

The opening of another year of club activities finds this country in the throes of a great war. With it the war has brought a great increase of interest in mathematics, and a greater realization of its importance. Among mathematicians and mathematics clubs there has been considerable emphasis on the applications of mathematics to the war effort. About half of the clubs whose reports have been received to date list in their programs topics related to this subject. The *Math X*, publication of the Washington Square College Mathematics Club, New York University, opens with the following paragraph. "The war effort has greatly

increased the need for training in mathematics with the emphasis on applied mathematics. There is a greater demand than can at present be supplied for young men and women who are prepared in the fields of mathematics, physics, engineering, and statistics."

UNDERGRADUATE PUBLICATIONS

The *Math X*, just quoted, sold out its complete edition of 400 copies. The papers included were as follows: *Königsberg's seven bridges* by Jack Moshman, *Mathematics and physical theories* by Frank Grace, *Non-dimensional variables* by Melvin Lax (including an interesting derivation of Van der Waal's equation connecting the pressure and volume of a gas at constant temperature), *A trick with numbers* by Marvin Forray and *An introduction to the calculus of finite differences* by Marvin Forray, *An extension of the max- and min- problem* by Harold Lewis, *Infinity and transfinite numbers* by S. M. Forman.

Other publications received were the Chapter News Letters of the *Kappa Mu Epsilon* chapters at Illinois State Normal University, Kansas State Teachers College at Emporia, and Mt. St. Scholastica College at Atchison, Kansas. Printed programs were received from Brown University, Central Michigan College, and the University of Kansas.

BIBLIOGRAPHIES

The report of the Mathematics Club of the University of Kansas included bibliographies for the following topics. Abstracts of the talks were also sent to the editor.

1. *The three great problems of ancient mathematicians* by Jean Bartz.
   Sanford, Vera. *A Short History of Mathematics.* Cambridge, Mass., Houghton Mifflin, 1930.
   Ball, Walter William. *Mathematical Recreations and Essays.* London, Macmillan, 1905.
   Cajori Florian. *A History of Mathematics.* Chicago, Open Court Pub. Co., 1928.
   Lovitt, William Vernon. *Elementary Theory of Equations.* New York, *Prentice-Hall*, 1939.

2. *Some problems in synthetic geometry* by O. C. Moots.
   Smith, D. E. *A History of Mathematics in America before 1900.* Chicago, Open Court Pub. Co., 1934.
   Cajori, Florian. *A History of Mathematics.* New York, Macmillan, 1931.
   Johnson, R. A. *Modern Geometry.* New York, Houghton Mifflin, 1929.
   Shively, Levi S. *An Introduction to Modern Geometry.* New York, J. Wiley and Sons, 1939.
   Coolidge, Julian L. *A History of Geometrical Methods.* Oxford, Clarendon Press, 1940.

3. *The history of the development of algebraic symbols* by Harwood Kolsky.
   Cajori, Florian. *History of Mathematical Notation.* Chicago, Open Court Pub. Co., 1928.
   Hill, G. F. *Development of Arabic Numerals in Europe.* Oxford, Clarendon, 1915.
   Smith, D. E. *Number Stories of Long Ago.* New York, Ginn and Co., 1919.
   Babylonian Math Sharks—2000 B.C. Sci. Am. 157: 311, N '37.
   G. A. Miller. *Implications in the history of math.* Sch. and Soc. 47: 275–7, F 26 '38.
   G. A. Miller. *Solution of equations by the ancients.* Sch. and Soc. 49: 178–9, F 11 '39.
   J. D. Buddhue. *Origin of our numerals.* Sci. Mo. 41: 490–500, D '35.
   G. A. Miller. *On the history of negative numbers.* Sci. Ns. 82: 517, N 29 '35.
   G. A. Miller. *Mathematical solution developed during more than three milleniums.* Sch. and Soc. 39: 211, F 17 '34.

G. A. Miller. *Our common numerals.* Sci. Ns. 78: 236–7, S 15 '33.
G. A. Miller. *Mathematical weakness of the early civilizations.* Sci. Mo. 33: 419–23, N '31.
L. C. Karpinski. *Descartes in the modern world.* Sci. Ns. 89: 150–2 F 17 '39.
W. Shepherd. *Fascination of figures.* Sci. Digest 7: 29–36, Ap '40.
R. D. Carmichael. *Number and clear thinking.* Sci. Mo. 41: 490–500 D '35.
Meanings of words in math. Science Ns. 92: sup. 8, N 29 '40.
S. Leacock. *Thru a glass darkly: Human thought in math. symbols.* Atlan. 158: 94–8, Jl '36.
R. C. Archibald. *Babylonian mathematics.* Sci. Ns. 70: 66–7, Jl 19 '29.
G. A. Miller. *Babylonian mathematics.* Sci. Ns. 72: 601–2, D 12 '30.

4. *The algebra of attributes* by Bruce Crabtree.
Birkhoff and MacLane. *A Survey of Modern Algebra.* New York, Macmillan, 1941. Chapter XI.
Birkhoff, Garrett. *Lattice theory* Am. Math. Soc. Colloq. Pub.

5. *Gambling* by Howard Barnett.
Culin, Stewart. *Gambling Games of the Chinese in America.*
Levinson, H. C., *Your Chance to Win.*
Proctor, R. A., *Chance and Luck.*
Editorials from *Detroit News,* 1922–3.
De King is daid! American Mercury, vol. 48; pp. 212–5 Oct. '39.
Gambling by wire. Christian Century, vol. 47; pp. 525–9, Ap 23 '30.
I can pick winners, but. Colliers, vol. 82; p. 24, Dec. 15 '28.
Change for the bettor; pari-mutuel. Colliers, vol. 105, pp. 52–3, June 15 '40.
You can't beat the races. Colliers, vol. 75, pp. 18–9, June 27 '25.
Betting and lotteries. Contemporary Reviews, vol. 145, pp. 715–21, June '34.
Pari-mutuel. Literary Digest, vol. 119, p. 38, Apr. 27, '35, vol. 116, p. 27, July 29 '33.
Poker permutations and combinations. Literary Digest, vol. 118, p. 33, July 21 '34.
Games of the moment (slot machines). New Outlook, vol. 165, p. 33, Apr '35.
Gamblers don't gamble. New Republic, vol. 99. p. 130, June 7 '39.

6. *The mechanics of rocket flight* by Howard Gadberry.
Oberth, Herrman. *Raketen zu den Interplaneteratum.*
R. Esnault-Pelterie. *L'Astronautique.*
Sanger, Eugen. *Raketenflugtechnic.*
Goddard, Robert H. *Liquid Propellant Rocket Development.*
Dresser, Peter. *Why Rockets Don't Fly.*

The following articles to be found in the *Journal of the American Rocket Society.*

Goodposture, R. *Laws of rocket motion.*
Africano, A. *Empirical rocket design formulas.*
Africano, A. *Velocity ratio efficiency.*
Sanger, E. *Rocket combustion motors.*
Wyld, James *Fundamentals of rocket motion.*
Dresser, Peter *Previewing the aerological rocket.*
Shesta, John *Thermal efficiency overemphasis.*
Shesta, John. *Theory of rocket operation.*

CLUB REPORTS 1941–42

Mathematics Society, University of Wisconsin at Milwaukee

Of the six papers presented, three were offered in competition for the annual Euler Prize in Mathematics. ***Pi, its derivation and transcendence*** by D. E. Mereen (the prize winner), ***Chance-probability*** by Herbert Reider, and ***"e"*** by Virginia Kubacki. The other three papers submitted dealt with subjects particularly suited to a nation at war. They were *Mathematics in artillery*

warfare by Professor W. E. Roth, *Navigation through the U. S. and Canada* by Miss Iverson (a discussion of methods of mapping with particular stress upon the navigation from an aeronautical standpoint), and *Simple mathematics in the present war* by Professor Morris Marden (a presentation of warfare problems set up for solution by the group). President, D. E. Mereen; Vice-President, R. E. Barr, Jr.; Secretary-Treasurer, Jean Petran; Faculty Adviser, Professor Marden.

Mathematics Club, Connecticut College

Members of our Mathematics Club gave short speeches on various subjects as follows: *Euclid* by Claire Peterson, *Quadrature of the circle* by Doris Kaske, *Trisecting an angle* by Dorothy Green, and *Squaring the cube* by Mary Powers. Our only outside speaker was Professor J. S. Frame of Brown University who spoke on *Mathematical problems in national defense*. At another meeting the Sophomores presented the play *The Flatlanders* taken from this MONTHLY. President, Doris Kaske; Vice-President, Alyce Watson; Secretary, Barbara Pilling; Treasurer, Mary Powers; Faculty Adviser, Professor Julia W. Bower.

Pi Mu Epsilon, Lehigh University

The Chapter held regular monthly meetings with mathematical programs, and sponsored one open lecture, given by Professor Otto Neugebauer, Brown University, on the subject *Origins of ancient mathematics*. The lecture was well attended and enjoyed by all who heard it. Officers for 1941–42 were: President, A. B. Brown, '42; Secretary, W. A. Eisele, '42; Treasurer, R. M. Maiden, '42; Faculty Adviser, Tomlinson Fort. Elected for 1942–43 were: President, M. G. Arsove, '43; Secretary, Stanley Caplan, '43; Treasurer, C. S. Bennett, '43.

Pi Mu Epsilon, University of Nebraska

Demonstrations of mathematical machines were held at a number of the meetings. A model of the Rigge curve tracing machine was explained by Joseph Martin, the slide rule by Vernon Vrana, a poor man's planetarium—an astronomical projector constructed and discussed by O. C. Collins, the harmonic analyzer by L. A. Rife, At a joint meeting with the local chapter of the American Institute of Electrical Engineers, Herbert Gaba spoke on the *Multiplication of complex numbers*. An initiation banquet, a quiz program and a spring initiation picnic completed the year's calendar. Director, D. L. Christensen; Vice-Director, Theodore Roesler; Secretary, Jean Hakanson; Treasurer, Dayle D. Rippe.

Kappa Mu Epsilon, Mount St. Scholastica College

At the opening initiation meeting of the Kansas Gamma Chapter ingenious tests were given to determine the candidates' fitness. The November meeting was a round table discussion on *Mathematics and national defense* with special emphasis on the role of the American college woman. A Christmas party was held in December, and in January a pop-corn sale, the proceeds of which were contributed for the support and continuance of the *Boletin Mathematico*. Two plays were presented by the chapter at an all school assembly: *Modern mathematics looks up its ancestors* by Marion Stark, and *A trip to infinity* by Tena Anders, a pledge. A meeting was held in February in honor of Lincoln, with emphasis upon his accomplishments in mathematics and upon those of his contemporaries. A "St. Patrick's" party was the second social event of the year, featuring an original skit *Irish medley*, with songs, recitations and dances. Pledges demonstrated their mathematical fitness for membership into *Kappa Mu Epsilon* on pledge day in April, and four new members were initiated at the May meeting. Officers for 1941–42 were: President Tartaglia, Bobbe Powers; Vice-President Cauchy, Mary Margaret Downs; Secretary Galileo, Margaret Mary Kennedy; Treasurer Napier, Mary Hughes; Faculty Sponsor, Sister Helen Sullivan, OSB. Elected for 1942–43 were President Proclus, Margaret Molley; Vice-President Cauchy, Mary Margaret Downs; Secretary Leibnitz, Virginia Meyers; Treasurer Bernouilli, Jane Schweizer.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Engineering Drawing. By D. E. Hobart. New York, D. C. Heath & Company, 1941. 7+430 pages. $2.75.

The text is one which will find a useful and welcomed place in the field. The book is well done with careful attention to conventions and accepted indications in standard practice. Particular commendation should be given the treatment of limit dimensioning and welding. Isometric drawing is clearly illustrated to expedite its understanding and use.

The chapter on fasteners is complete with ample tables to cover student needs. Much has been made of that portion on charts and graphs.

In construction drawing, architectural drawing perhaps purposely was neglected but the treatment of structural steel is highly commended.

In intersections and developments there is given the needed descriptive geometry for a full understanding of the subject and the timely treatment of surfaces of double curvature will prove of value to those interested in lofting.

In general the book is concisely expressed and clearly written. The theory is ably presented and consideration given to detail shows the author completely familiar with the problems which arise in teaching the subject. For greater choice or needed variety for those using the book in successive years the problem content could be expanded.

R. W. BOCKHORST

An Introduction to Analytical Geometry and Calculus. By T. K. Raghavachari. Mount Road, Madras, Oxford University Press, 1941. 20+192 pages. Rs 2.

This little book was written specifically as a textbook in the intermediate courses of South Indian Universities and without doubt adequately serves the purpose for which it was intended. There seems, however, to be little place in the American colleges or universities for such a treatment of analytic geometry and calculus as it is too long for a course of three hours and too short for six. It could possibly be used in a one-year cultural course by giving a very short treatment of trigonometry between the analytic geometry and calculus. The author is an excellent expositor and if he could maintain his clarity throughout a longer book covering what Americans consider standard he would have a book that would undoubtedly find its place among the textbooks of this country.

The book is pocket size and contains 192 pages about equally divided be-

tween geometry and calculus. Only the straight line and circle are included in analytic geometry but functions, graphs, and tangents to curves are discussed under calculus. The part on calculus, though short, can by no means be considered a "calculus made easy" in the bad sense of the phrase, as more than usual care is taken to tell the truth. For example, the area under a curve is defined (not proved) to be the limit of the sum of areas of rectangles and moreover the existence of this limit is made quite plausible.

There are of course omissions from standard courses. Conic sections are not mentioned and the definite integral is applied only to areas under or between curves and to volumes of revolution. On the other hand, the treatment of the straight line is most exhaustive. A somewhat startling inclusion is an extension of directed distances to directed areas; "Areas will therefore be positive if the perimeter is described in the counter-clockwise direction and negative if clockwise." The author should probably not be blamed for his "proof" that $dx=\Delta x$.

It is with some regret that such an attractive book both in appearance and exposition cannot be recommended for use in America.

J. F. RANDOLPH

Tables of Natural Logarithm. Volume IV. Contains the Decimal Numbers from 5.0001 to 10.0000. (Prepared by the Federal Works Agency Projects Administration for the City of New York. Conducted under the Sponsorship of the National Bureau of Standards.) New York, Work Projects Administration, 1941. 22+506 pages. $2.00.

This is Volume IV of a series of four volumes of natural logarithms. It contains the natural logarithms of decimal numbers from 5 to 10 at intervals of 0.0001.

The corrections of errors in the Wolfram Tables, and the explanations of the procedure for direct and inverse interpolation are repeated from the earlier volumes. The arrangement of the page and the safeguards for a high degree of accuracy are the same as in the preceding volumes of the series.

VIRGIL SNYDER

TechniData Hand Book. By Edward L. Page. New York, The Norman W. Henley Publishing Company, 1942. 64 pages. Spiral binding, $1.00. Cloth binding, $1.50.

This book contains a well-selected group of formulas, definitions, laws, figures, tables, etc., from mathematics, mechanics, physics, chemistry, and engineering, in a form convenient for quick reference. In many places the typography is not clear, exponents being particularly hard to read. Quite a few obvious errors were noted, the most serious of which occurred in the solution of the general cubic equation. It seems especially unfortunate that so many errors should occur in a handbook.

N. G. GUNDERSON

Brief Course in Analytics. By M. A. Hill and J. B. Linker. New York, Henry Holt and Company, 1940. 9+204 pages. $1.36.

This elementary text in Analytic Geometry, though designated brief, covers all the topics usually given in a first course in analytics. It includes the following chapters: Basic Definitions and Theorems, Equations and Loci, Straight Lines, Special Equations of the Second Degree, General Equation of the Second Degree, Equations in Other Forms, Elements of Solid Analytic Geometry.

The explanations are clear but concise. The arrangement of topics is good. "Many problems have been included in the text, some of which are quite easy, others difficult. By solving the former, the student will gain confidence in himself and will be able to apply his knowledge in solving the latter."

Answers are given for the odd numbered problems. The text should satisfy both the student and the instructor.

R. P. STEPHENS

College Algebra. By C. H. Sisam. New York, Henry Holt and Company, 1940. 12+395 pages. $1.90.

The author of an earlier work on Analytic Geometry has here produced a new College Algebra. The outstanding feature seems to be the flexibility permitted in using the text to meet college algebra classes of varying ability and different high-school background. Sufficient material is presented on the ordinary elementary algebraic processes to make the text suitable for the high-school graduate who has been exposed to but one year of algebra. The insertion of a section on business mathematics and some mathematical tables is a unique feature. The exercises seem to be sufficiently numerous to permit instructors to select those best suited to the needs of the individual students.

A. V. KOZAK

Intermediate Algebra for College Students. By T. S. Peterson. New York, Harper and Brothers, 1942. 8+358 pages. $1.85.

This well written book covers those topics usually found in intermediate algebra textbooks; to quote, "the author has purposely neglected to stress certain principles which are more effectively studied in college algebra and which have little value for the student who does not intend to continue work in mathematics." In accordance with this purpose, about three-fourths of the book is devoted to algebra through simultaneous quadratic equations, and about fifty pages are devoted to ratio, variation, the binomial theorem, logarithms, and progressions. A number of sections are marked "optional." The rigor is sufficient for the purposes intended.

There are a number of features which may appeal to teachers. Word statements of rules, usually approached intuitionally, are very prominentlydisplayed; symbolic statements receive less emphasis. Common errors, and points which the student might overlook otherwise, are brought to his attention through

"notes." A great deal of emphasis is placed upon the checking of solutions; there is a good note on page 295 on estimating the reasonableness of logarithmic solutions.

The "stated problem" is stressed throughout the book, the recommended procedure consisting of analysis, solution, statement of the answer in words, and check. The familiar tabulated forms of the secondary school textbooks are utilized in the analysis of problems, and certain problems are diagrammed as well.

There are many examples of illustration; many students who wish to acquire facility in the use of elementary algebraic operations could use this book without additional instructional assistance. The exercises, and the review lists at the end of each chapter, contain an ample number of problems for classwork as well as for outside preparation; all of the problems are of familiar types and they are well chosen. Answers are given to most of the odd numbered problems; in the sample selected for solution by this reviewer no errors were detected.

Some teachers may think that too much attention is given to "transposition" and that too little attention is given to the operation of adding the same quantity to each member of an equation. A suggestion, perhaps a very minor one, for future editions of the book, is that some principle of order be adopted for the entries in the tables printed on pages 216, 217, 255 and 256; most teachers probably would prefer the order used in the tables printed on pages 219 and 223 to that used in the table on page 220.

The general appearance of the book is attractive; the only detected printing error is an obvious and harmless one on page 31.

C. W. MUNSHOWER

An Introduction to the Theory of Newtonian Attraction. By A. S. Ramsey. Cambridge, The University Press; New York, The Macmillan Company, 1941. 9+184 pages. $2.50.

This is a textbook setting forth the elementary groundwork of the classical theory of gravitational attraction and potential. In this purposely limited respect it is a welcome addition to the textbook literature in English. It is the last of a series of the well-known books on Mechanics which Mr. Ramsey has written in recent years, and he is to be congratulated on his perseverance to so complete and successful a conclusion of his task.

The first chapter is concerned with some "Preliminary Mathematics" including the Laplacian in curvilinear coordinates and the convergence of volume integrals. There are some remarks on "vectorial methods," but the vector analysis is not employed in this book. In the remaining six chapters one finds in order: the consideration of the usual elementary theorems and problems on the gravitational attraction and potential at points outside and inside attracting continuous matter (II, III); the theorems of Laplace, Poisson, and Gauss (IV) and of Green (V); harmonic functions of integral degree (VI); and the attraction of

ellipsoids, with a few brief final remarks on the equilibrium of a rotating fluid mass, (VII).

The author's style is pleasingly direct; he writes with an admirable economy of language. The physical ideas are everywhere carefully presented, but in certain places important questions should have received, in the reviewer's opinion, more attention to mathematical rigor. The book is well illustrated with over fifty diagrams, and the usual long sets of valuable College examination problems have been selected with care. There is an occasional and discriminating use of geometrical argument. Although there are a few incidental references scattered throughout the book there is no bibliography, which is most unfortunate. Moreover, this subject has a long and glorious history and the author's bald biographical footnotes are hardly the way of meeting the responsibility of referring to it.

Since formal courses in this subject at the undergraduate level for which this book is expressly written are apt to be rare, it should be stated that serious students in mathematics and physics will profit from reading this book independently during or immediately following the first course in analytic mechanics.

S. G. HACKER

Tools. A Mathematical Sketch and Model Book. By R. C. Yates. Baton Rouge, Louisiana State University, R. C. Yates, 1941. 194 pages. $1.60.

This interesting and different book "has been designed especially for college students who are prospective students of mathematics" and treats of geometrical instruments, their use and their equivalence. Freshman work in algebra, trigonometry, and analytics is the only mathematics presupposed.

The nature of the material presented is best seen from the titles of the eleven sections of the book. These are: The Straightedge and Modern Compasses (Modern Geometry), Dissection of Plane Polygons, The Compasses (Geometry of Mascheroni), Folds and Creases (Geometry of Paper Folding), The Straightedge (Synthetic Projective Geometry), Line Motion Linkages (How to Draw a Straight Line), The Straightedge with Immovable Figure (Geometry of Poncelet-Steiner), The Assisted Straightedge, Parallel and Angle Rulers, Higher Tools and Quartic Systems, and General Plane Linkages.

Not presuming to be complete, each section is preceded by a bibliography for further reading and consists of explanatory text, together with many plates which are to be worked out by the student in laboratory periods.

The present book is a loose-leaf affair, printed by the author, and hence not of a very durable nature. However, much of unusual interest is to be found here, which will be valuable to any teacher of mathematics and will make excellent supplementary class material.

R. A. HARRISON

NEW BOOKS RECEIVED

General Trade Mathematics. By E. P. Van Leuven. New York, McGraw-Hill Book Co., 1942. 10+575 pages.

Logarithms, Trigonometry, Statistics. First Year College Mathematics. By H. R. Cooley, P. H. Graham, F. W. John, and A. Tilley. New York, McGraw-Hill Book Co., 1942. 12+280 pages. $2.00.

A Short Course in Tensor Analysis for Electrical Engineers. By G. Kron. New York, John Wiley and Sons, Inc., 1942. 15+250 pages. $4.50.

The Fundamental Principles of Mathematical Statistics with Special Reference to the Requirements of Actuaries and Vital Statisticians and an Outline of a Course in Graduation. By H. H. Wolfenden. New York, Actuarial Society of America, and the Macmillan Company of Canada, 1942. 15+379 pages.

Introduction to the Theory of Relativity. By P. G. Bergmann. New York, Prentice-Hall, Inc., 1942. 16+287 pages. $4.50.

Tables of Natural Logarithms, Volume IV. Logarithms of the decimal numbers from 5.0000 to 10.0000. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York. Conducted under the sponsorship of the National Bureau of Standards, 1941. 22+510 pages. $2.00.

Elements of Spherical Trigonometry. By J. E. Thompson. New York, D. Van Nostrand Company, 1942. 12+144 pages. $1.65.

An Outline of College Algebra. By G. E. Moore. New York, Barnes and Noble, Inc., 1942. 224 pages. $1.00.

Essentials of Astronomy. By J. C. Duncan. New York, Harper and Bros., 1942. 181 pages. $1.85.

Spherical Trigonometry with Naval and Military Applications. By L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill Book Co., Inc., 1942. 13+163 pages. $1.50.

Mathematical Dictionary. By G. James and R. C. James. Van Nuys, California, The Digest Press, 1942, 5+259+22 pages. $3.00.

First Year College Mathematics. By C. C. Richtmeyer and J. W. Foust. New York, F. S. Crofts and Company, 1942. 11+461 pages. $3.25.

Industrial Statistics. Statistical Technique Applied to Problems in Industrial Research and Quality Control. By H. A. Freeman. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1942. 9+178 pages. $2.50.

Contributions to the Calculus of Variations, 1938–1941. Theses submitted to the Department of Mathematics of the University of Chicago and Bibliography. Chicago, University of Chicago Press, 1942. 7+527 pages.

Mathematics of Business and Finance. By W. B. Dyees and R. C. Gilmore. (Including Compound Interest and Annuity Tables by F. C. Kent and M. E. Kent.) New York, McGraw-Hill Book Co., 1942. 10+221+8+214 pages. $3.50.

Basic College Mathematics. (A General Introduction.) By C. W. Munshower and J. F. Wardwell. New York, Henry Holt and Co., 1942. 11+612 pages. $3.20.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 531. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle?

E 532. *Proposed by V. Thébault, San Sebastián, Spain*

Find a perfect square whose digits form one of the permutations of five consecutive digits.

E 533. *Proposed by N. A. Court, University of Oklahoma*

Prove that, if an orthocentric group of points occurs as a section of an orthocentric group of lines, then the plane of section is perpendicular to one of the lines.

E 534. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

E 535. *Proposed by A. H. Stone, Institute for Advanced Study*

Let A', B', C' be three points on the circumcircle of a triangle ABC, whose Simson lines with respect to ABC all meet in a point, O. Prove that the Simson lines of A, B, C, with respect to the triangle $A'B'C'$, concur at the same point O.

E 511 [1942, 195]. *Proposer's Correction*

After "Doppler effect," the problem should read as follows:

The train of sound waves of maximum apparent pitch *leaves* the bomb at the elevation at which the component of the bomb's velocity in the direction of the observer is a maximum. Find this angular elevation, and show that it approaches two-thirds of the initial elevation if the initial elevation is small.

SOLUTIONS

A Magic Square of Triangular Numbers

E 496 [1941, 699]. *Proposed by R. V. Heath, Wall St., New York City*

What is the smallest value of n for which the n^2 triangular numbers 0, 1, 3, 6, 10, $\cdots$, $\frac{1}{2}n^2(n^2-1)$ can be arranged to form a magic square?

Partial solution by the proposer

Such an arrangement can certainly be found for $n=8$.

A magic square is said to be doubly-magic if its sums remain uniform when all its numbers are squared. Clearly, the magic property will still be retained if each of the original numbers is subtracted from its square. The resulting numbers are all even, and their halves are the triangular numbers; in fact each original number r leads to $\frac{1}{2}r(r-1)$. Here, for instance, is a doubly-magic pandiagonal square (cf. Rouse Ball, *Mathematical Recreations and Essays*, 11th edition, p. 212, Fig. xxviii) along with the corresponding magic square of triangular numbers:

16	41	36	5	27	62	55	18
26	63	54	19	13	44	33	8
1	40	45	12	22	51	58	31
23	50	59	30	4	37	48	9
38	3	10	47	49	24	29	60
52	21	32	57	39	2	11	46
43	14	7	34	64	25	20	53
61	28	17	56	42	15	6	35

120	820	630	10	351	1891	1485	153
325	1953	1431	171	78	946	528	28
0	780	990	66	231	1275	1653	465
253	1225	1711	435	6	666	1128	36
703	3	45	1081	1176	276	406	1770
1326	210	496	1596	741	1	55	1035
903	91	21	561	2016	300	190	1378
1830	378	136	1540	861	105	15	595

(In this last square, each of the eight rows, eight columns, and two main diagonals has the sum 5460.)

But it remains possible that a smaller set of triangular numbers might form a magic square without the corresponding natural numbers forming a magic square. Moreover, it has never been satisfactorily proved that there is no doubly-magic square of order 7.

Partition of a Triangle

E 497 [1941, 699]. *Proposed by V. Thébault, San Sebastián, Spain*

The sides of a triangle $A'B'C'$, of constant size, remain parallel to those of a fixed triangle ABC, and form with it three more triangles and three pentagons. Show that the position of $A'B'C'$ which minimizes the sum of the areas of these three triangles makes the areas of the three pentagons all equal.

Solution by Howard Eves, Chattanooga, Tenn.

Let us designate the three triangles (all similar to ABC) by $T(A)$, $T(B)$, $T(C)$, where $T(A)$ corresponds to the vertex A in the obvious manner. First, let $A'B'C'$ move so that $T(A)$ remains constant in area. This means that the side $B'C'$ slides along a fixed line parallel to BC. In order for the area $T(B)+T(C)$ to be a minimum, we must have $T(B)=T(C)$. Treating $T(B)$ and $T(C)$ similarly, we see that, in order for the area $T(A)+T(B)+T(C)$ to be a minimum, we must have $T(A)=T(B)=T(C)$. This guarantees that the parallelograms AA', BB', CC' be all equal in area (since in pairs they have equal bases and altitudes); and this in turn implies equality of area for the three pentagons.

Note. Triangles ABC and $A'B'C'$ have a common centroid, since the lines AA', BB', CC' are medians of both.

Also solved by the proposer.

A Convergent Sequence

E 498 [1941, 699]. *Proposed by E. C. Kennedy, Texas College of Arts and Industries*

Consider the relation

$$T_{n+1} = \sqrt{\frac{k+T_n}{2-T_n}}.$$

What is the largest value of k such that the sequence $\{T_n\}$, for a suitable range of values of T_0, converges to a positive number? What is this number?

Solution by P. D. Thomas, Southeastern State College, Durant, Okla.

Assume that $\{T_n\}$ has a positive limit, T. Then

$$T^2 = (k+T)/(2-T),$$

or

$$T^3 - 2T^2 + T + k = 0. \tag{1}$$

This equation has no positive root if k is positive, but at least one otherwise. (If $-4/27<k<0$, it has three.) Thus the required largest value of k is 0, and then $T=1$.

(Cf. No. 379 on p. 380 of the *National Mathematics Magazine*, April, 1941.)

Also solved by the proposer.

Editorial note. If $-2<k\leqq 0$, the sequence $\{T_n\}$ converges to the largest root of (1), provided T_0 lies between this root and 2.

Centers of Similitude

E 499 [1941, 699]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Two intersecting circles (A) and (B) have centers mutually external. Two other circles (C) and (D), orthogonal to (A) and (B) respectively, are drawn through the points of intersection. Show that the two common tangents of (C) and (D) are concurrent with the two common tangents of (A) and (B).

Solution by W. F. Clarke, San Jose, Calif.

Let E and F be the intersections of (A) and (B), and P the point where their common tangents meet AB. By the requirements of the problem, (C) and (D) must pass through E and F, with EC perpendicular to EA, and ED to EB. Clearly, angles AEB and CED have a common bisector, which meets AB at K, the harmonic conjugate of P with respect to A and B. Since P and K are the external and internal centers of similitude of (A) and (B), it follows that EP is the external bisector of both angles AEB and CED. Thus P is the harmonic conjugate of K with respect to C and D, whence the common tangents of (C) and (D) must also pass through P.

Also solved by Howard Eves, P. D. Thomas, and the proposer.

Editorial Note. The result can be obtained more rapidly as follows. With respect to the circle through E with center P, (A) inverts into (B). Therefore (C) inverts into (D), and the tangents from P to (C) touch (D) also.

Differences of Factorials

E 488 [1942, 377]. *Generalization by H. W. Becker, Vallejo, Calif.*

Let $\{u_k\}$ be any sequence, and $v_k=(E-x)^k u_0$. Then

$$(E+x)^n v_0 = u_n.$$

Proof. Since $E^k v_0=(E-x)^k u_0$, we have

$$(E+x)^n v_0 = \sum_{r=0}^{n}\binom{n}{r}x^r E^{n-r}v_0 = \sum_{r=0}^{n}\binom{n}{r}x^r(E-x)^{n-r}u_0$$
$$= (E-x+x)^n u_0 = u_n.$$

The case when $v_k=k!$ suggests the notation

$$k!_x = (E+x)^k 0! = k!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}\right),$$

whence

$$0!_x = 1, \qquad k!_x = k(k-1)!_x + x^k,$$

and $k!_0$ is the ordinary factorial. The above theorem then gives

$$(E-x)^n 0!_x = n!.$$

Problem E 488 itself is included, $k!_1$ being the "super-factorial," and $k!_{-1}$ the "sub-factorial."

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4045. *Proposed by A. M. Glicksman, The Bronx High School of Science*

Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right).$$

4046. *Proposed by Otto Dunkel, Washington University*

Show that Euler's constant γ is given by

$$\gamma = 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots\right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}.$$

4047. *Proposed by Theodore R. Running, Ann Arbor, Mich.*

Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, and area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974
.	.	.	.

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

4048. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of six digits $N = abcdef$ such that the product NN' is a perfect square, where $N' = defabc$.

4049. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron $ABCD$ the straight lines joining the centroid with the circumcenter of the triangles of the faces cut the respective radical planes of the circumsphere and the spheres with the medians of $ABCD$ as diameters in four points of the same plane.

SOLUTIONS

Inequalities for Intervals

3949 [1940, 182]. *Corrected. Proposed by P. Turán, Budapest, Hungary*

Given the angles $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$ with the common initial line Ox, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{n(n-1)/2+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$, or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$.

Solution by the Proposer

We consider in turn three cases:

I. If $\min |\phi_\mu - \phi_\nu| \geqq \pi/2^\lambda$, $\lambda = n(n-1)/2$, we may take $\beta = \pi/2^{\lambda+1}$.
II. If $\max |\phi_\mu - \phi_\nu| \leqq \pi/2$, we may take, for example, $\beta = 5\pi/8$.
III. If $\min |\phi_\mu - \phi_\nu| < \pi/2^\lambda$, and $\max |\phi_\mu - \phi_\nu| > \pi/2$, let $\phi_\mu - \phi_\nu = \Delta_{\mu\nu}$, $\mu > \nu$.

We now arrange these differences in monotonic increasing order, and for simplicity we use simple subscripts $0 < \Delta_1, \Delta_2, \cdots, \Delta_\lambda < 2\pi$. The conditions here evidently mean that

$$\Delta_1 < \pi/2^\lambda = 2\pi/2^{\lambda+1}, \qquad \Delta_\lambda > 2^\lambda \pi/2^{\lambda+1};$$

and it is obvious that there exists an integer r such that

$$1 \leqq r < \lambda, \qquad \Delta_r \leqq 2^r \pi/2^{\lambda+1}, \qquad \Delta_{r+1} \geqq 2^{r+1}\pi/2^{\lambda+1}.$$

This means that for *no* k and ν is

$$2^r \pi/2^{\lambda+1} < |\phi_k - \phi_\nu| < 2^{r+1}\pi/2^{\lambda+1},$$

and hence a suitable choice is

$$\beta = 2^r \pi/2^{\lambda+1}.$$

Tetrahedral Polars

3993 [1941, 273]. *Proposed by N. A. Court, University of Oklahoma*

A variable plane passing through a fixed point of the face ABC of the tetrahedron $DABC$ meets the edges DA, DB, DC in the points P, Q, R. Show that the locus of the point U common to the three planes PBC, QCA, RAB is a cone of the second degree.

I. *Solution by H. S. M. Coxeter, University of Toronto*

Let (x, y, z, t) be the barycentric coordinates of U, referred to the tetrahedron $ABCD$, so that the points P, Q, R are

$$(x, 0, 0, t), \quad (0, y, 0, t), \quad (0, 0, z, t), \qquad (t \neq 0).$$

The condition for these to be coplanar with the fixed point $(f, g, h, 0)$ is

$$\begin{vmatrix} x & 0 & 0 & 1 \\ 0 & y & 0 & 1 \\ 0 & 0 & z & 1 \\ f & g & h & 0 \end{vmatrix} = 0,$$

or $fyz+gzx+hxy=0$. Thus the locus of U is a quadric cone with vertex D.

II. *Solution by the Proposer.*

Let s be the trace of the variable plane PQR in the face ABC, and S the harmonic pole of the line s for the triangle ABC.

The harmonic pole M of the plane PQR for the tetrahedron $DABC$ lies on the line DS, and also in the plane BCP' projecting from BC the harmonic conjugate P' of P to the vertices D, A. Hence the plane BCP meets the line DS in the harmonic conjugate U of M for the pair of points D, S. Similarly the planes CAQ and ABR pass through U.

If the trace s remains fixed and the variable plane PQR revolves about s, the point U describes the fixed line DS.

The line s passes through the fixed point L given in the plane ABC. As the line s revolves about L, the point S describes a conic (L) circumscribed about the triangle ABC (J. J. Mathieu, *Nouvelles Annales de Mathématiques*, 1865, p. 407). Consequently the locus of the point U is the cone obtained by projecting the conic (L) from the vertex D.

Remark. We have incidentally proved the proposition:

If a plane revolves about a fixed point in a face of a tetrahedron, the harmonic pole of the variable plane for that tetrahedron describes a cone of the second degree having for vertex the vertex of the tetrahedron opposite the face containing the fixed point.

Indeed, the cone $D(L)$ is also the locus of the point M.

Solved also by G. B. Huff, W. T. Short, C. E. Springer and P. D. Thomas.

Editorial Note. The remaining solutions are analytic, and Huff stated that the same analysis applies to any number of dimensions greater than one. The theorem by Mathieu used by the proposer is easily proved. For, if the plane of P, Q, R cuts the sides BC, CA, AB in A_1, B_1, C_1 on the straight line s, let A_1', B_1', C_1' be the respective harmonic conjugates of the last three points with respect to the corresponding pair of vertices of triangle ABC. Then AA_1', BB_1', CC_1' intersect in S, the pole of s with respect to triangle ABC. If s rotates in the plane of ABC about the fixed point L, the points A_1, B_1, C_1 describe projective ranges on their respective bases; and we have the projective ranges $(A_1')\bar{\wedge}(A_1)\bar{\wedge}(B_1)\bar{\wedge}(B_1')$. It follows that $A(A_1')$, $B(B_1')$, $C(C_1')$ are projective pencils, and that S describes a conic (L) passing through the vertices of ABC. If AL cuts BC in A_0, then AA_0' is the tangent at A, where A_0, B, A_0', C is a harmonic set of points. If B_0', C_0' are similarly defined, then the three tangents

AA_0', BB_0', CC_0' are such that A_0', B_0', C_0' lie on a straight line l which is the polar of L with respect to ABC. If L lies on a side, say AB, then C_1 is the fixed point L for all positions of s, and C_1' is also a fixed point, and S describes the straight line CC_1'. The conic degenerates into the pair of straight lines of AB and CC_1' which is of the hyperbolic type, except when L is at the midpoint of BC, and then we have the parabolic type with AB and CC_1' as parallel straight lines.

Conversely, if any conic passes through the vertices of triangle ABC, there exists a point L by means of which the conic is generated in the above manner. For, let the tangents at the vertices cut the opposite sides in A_0', B_0', C_0'. Then the triangle and the three tangents form a degenerate hexagon inscribed in the conic whose opposite sides meet respectively in A_0', B_0', C_0', and these three points lie on a straight line l by Pascal's theorem. The polar of A_0' with respect to the conic is the straight line through A cutting BC in A_0 so that B, A_0, C, A_0' is a harmonic set; hence AA_0, BB_0, CC_0 must meet in a point L which is the pole of l with respect to the conic and also with respect to the triangle ABC; and L is the desired point.

We now determine the regions of the plane for L which give the different types of the conic (L). If the conic (L) has a point at infinity S_∞, the polar of this point with respect to ABC is the straight line s_∞ passing through L, and if it has two distinct points at infinity, the intersection of their polars is L. We consider therefore the envelope γ of the one-parameter family of polars with respect to ABC of points on the line at infinity. The envelope γ must be a conic; for, if L is any point of the plane, the conic (L) cuts the line at infinity in two points, and hence there are precisely two tangents from L to γ, which may coincide and then L is on γ. We show next that the conic γ is tangent to the sides of ABC at their midpoints, and this determines it as an ellipse whose center is at G, the centroid of ABC. The polar with respect to ABC of the point at infinity on AB is AB, and hence this side is tangent to γ. At any point C_1 of AB other than the midpoint M_c the conic (C_1) is of the hyperbolic type and there are two distinct tangents from C_1 to γ, one of which is AB; but for the midpoint M_c the conic (M_c) is parabolic and the two tangents from M_c to γ coincide in AB, with the limit point M_c as the point of contact of AB. We now have the result:

If L is inside the ellipse γ the conic (L) is an ellipse; if L is on γ the conic (L) is a parabola; if L is inside ABC but outside of γ the conic (L) is a hyperbola with one branch passing through the vertices of ABC; if L is outside of ABC then (L) is a hyperbola with one branch through one vertex of ABC and the other branch through the remaining two vertices.

If L is at the centroid G of ABC the conic (G) is an ellipse with center G tangent to the sides of the triangle complementary to ABC at their midpoints A, B, C. The two ellipses γ and (G) have other important properties.

This completes the determination of the form of the cone for the various positions of L.

There are three polars of a given point $P(y_1, y_2, y_3, y_4)$ with respect to a given tetrahedron $A_1A_2A_3A_4$, whose equations are

$$(1)\quad \sum \frac{y_i}{x_i} = 0, \qquad (2)\quad \sum_{i \neq j} \frac{y_i y_j}{x_i x_j} = 0, \qquad (3)\quad \sum \frac{x_i}{y_i} = 0, \qquad i, j = 1, 2, 3, 4.$$

The third polar, or polar plane of P, has the following property: If P' is a point on the polar plane of P, and the straight line joining P and P' cuts the faces of the tetrahedron, the surface $x_1x_2x_3x_4 = 0$, in the points Q_1, Q_2, Q_3, Q_4, then we have the harmonic relation

$$\frac{4}{PP'} = \sum \frac{1}{PQ_i}.$$

This result and also corresponding relations for the first and second polars are easily obtained by requiring the sum of the coordinates of each finite point to be a constant, say unity.

Partial Differential Equation

3995 [1941, 273]. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the partial differential equation

$$y(x + y)z_{xx} - (x^2 - y^2)z_{xy} - x(x + y)z_{yy} + (x - y)(z_x + z_y) = 0.$$

Solution by H. W. Bailey, University of Illinois

This equation is readily solved by a Laplace transformation (e.g., Morris and Brown, *Differential Equations*, Prentice-Hall, New York, 1935, p. 340 ff.). The Monge auxiliary equation

$$(x + y)[ydy^2 + (x - y)dydx - xdx^2] = 0$$

has solutions $x^2+y^2=a$, $x-y=b$. On making a transformation of variables $u=x^2+y^2$, $v=x-y$, the differential equation becomes $z_{uv}=0$ which has the general solution $z=\phi(u)+\psi(v)$. Hence, in terms of x and y, the general solution is

$$z = \phi(x^2 + y^2) + \psi(x - y).$$

Solved also by G. W. Petrie, P. D. Thomas, A. K. Waltz, and C. L. Weaver. The proposer stated that the solution could be obtained by the above change of variables to u and v.

Summation of Series

3996. [1941, 341] *Proposed by Elbert H. Clarke, Hiram College*

Sum the series

$$\sum_{n=1}^{\infty} [(n - 1)k]!/(nk)!,$$

where k is any integer greater than unity.

Solution by the Proposer

The general term, $1/(nk+1)(nk+2)\cdots(nk+k)$ can be separated into its partial fractions by the usual methods, and it is easily discovered that we have a common factor $1/(k-1)!$ and that the successive fractions have numerators which are the binomial coefficients in the expansion of $(1-1)^{k-1}$. This suggests that the original sum can be expressed as $f(1)$ where

$$f(x) = \frac{1}{(k-1)!}\left[\frac{x}{1} - \binom{k-1}{1}\frac{x^2}{2} + \binom{k-1}{2}\frac{x^3}{3} - \cdots\right.$$

$$(1) \qquad + (-1)^{k-1}\binom{k-1}{k-1}\frac{x^k}{k} + \frac{x^{k+1}}{k+1} - \binom{k-1}{1}\frac{x^{k+2}}{k+2}$$

$$\left. + \binom{k-1}{2}\frac{x^{k+3}}{k+3} - \cdots + (-1)^{k-1}\binom{k-1}{k-1}\frac{x^{2k}}{2k} + \cdots\right]$$

each cycle of terms carrying the same binomial coefficients while the powers of x rise steadily in step with the regular progression of the denominators. It is seen at once that the derivative $f'(x)$ has a simple form in which the factor $(1-k)^{k-1}$ occurs multiplied by a geometric series of ratio x^k. In symbols,

$$f'(x) = \frac{1}{(k-1)!}\,\frac{(1-x)^{k-1}}{1-x^k}.$$

Hence the original series may be written

$$(2) \qquad \sum_{n=1}^{\infty}\frac{[(n-1)k]!}{[nk]!} = \frac{1}{(k-1)!}\int_0^1 \frac{(1-x)^{k-2}}{1+x+x^2+\cdots+x^{k-1}}\,dx.$$

Again using partial fractions, the integrand may be written $B_1/(x-\alpha_1) + B_2/(x-\alpha_2) + \cdots + B_j/(x-\alpha_j) + \cdots + B_{k-1}/(x-\alpha_{k-1})$ where α_j is one of the complex kth roots of 1. By the usual reductions it is found that $B_j = -\alpha_j(1-\alpha_j)^{k-1}/k$.

Hence the original infinite series is reduced to a finite sum

$$\frac{1}{k!}\sum_{j=1}^{k-1}(-\alpha_j)(1-\alpha_j)^{k-1}\log\left(\frac{1-\alpha_j}{-\alpha_j}\right).$$

When this is translated in the ordinary manner into a sum of real and imaginary terms it is easy to show that the imaginary terms cancel out and the real terms double up, all of them if k is odd, and all but one if k is even. If we use $\phi\,(k, j)/2^k$ to stand for the expression to be summed

$$\phi(k, j) = \left(\sin\frac{\pi j}{k}\right)^{k-1}\left[\cos\frac{\pi(k-2j)(k+1)}{2k}\log\left(2\sin\frac{\pi j}{k}\right)\right.$$

$$\left. + \frac{\pi}{2k}(k-2j)\sin\frac{\pi(k-2j)(k+1)}{2k}\right],$$

and the sum of the infinite series with which we started is

$$\sum_{n=1}^{\infty} \frac{[(n-1)k]!}{[nk]!} = \frac{2^k}{k!} \sum_{j=1}^{(k-1)/2} \phi(k, j),$$

if k is odd, and

$$\frac{2^{k-1} \log 2}{k!} + \frac{2^k}{k!} \sum_{j=1}^{(k-1)/2} \phi(k, j),$$

if k is even.

Series for $k=2, 3, 4, 5$, are given for reference.

$$k = 2: \qquad \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \log 2,$$

$$k = 3: \qquad \frac{1}{1\cdot 2\cdot 3} + \frac{1}{4\cdot 5\cdot 6} + \cdots = \frac{1}{4}\left[\frac{\pi\sqrt{3}}{3} - \log 3\right],$$

$$k = 4: \qquad \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots = \frac{1}{4}\left[\log 2 - \frac{\pi}{6}\right],$$

$$k = 5: \quad \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{1}{6\cdot 7\cdot 8\cdot 9\cdot 10} + \cdots = \frac{1}{48}\left[\log 2 - \frac{1+\sqrt{5}}{2} \log 20 + \sqrt{5} \log (5 - \sqrt{5}) + \frac{\pi\sqrt{2}}{10}(\sqrt{25 + 11\sqrt{5}} - 3\sqrt{25 - 11\sqrt{5}})\right].$$

Editorial Note. The equality (2) results from the Abel theorem given in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, p. 378. The summation of series of a similar form is given in the solution of 2907 [1923, 206].

Slopes and Curvatures

3999 [1941, 409]. *Proposed by G. B. Van Schaack, Michigan State College*

Let $f(x)$ be a polynomial of degree n with n distinct real roots x_i, $(i=1, 2, \cdots, n)$. Let λ_i be the reciprocal of the slope of the curve $y=f(x)$ at $x=x_i$. Let ρ_j, $(j=1, 2, \cdots, n-1)$, be the algebraic radius of curvature of the curve at the critical point of the curve which lies between x_j and x_{j+1}. (a) Show that if $n>1$, then $\lambda_1+\lambda_2+\cdots+\lambda_n=0$. (b) Show that if $n>2$, then $\rho_1+\rho_2+\cdots+\rho_{n-1}=0$.

Solution by Paul Brock, Student, Brooklyn College

We may consider the coefficient of the highest power of x in $f(x)$ to be unity, and then

$$f'(x_1) = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n),$$

where $n>1$ and no two x_i's are equal. We shall prove that the following sum is zero

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = \frac{1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)\cdots(x_1-x_n)} + \frac{1}{(x_2-x_1)(x_2-x_3)(x_2-x_4)\cdots(x_2-x_n)} + \frac{1}{(x_3-x_1)(x_3-x_2)(x_3-x_4)\cdots(x_3-x_n)} + \cdots + \frac{1}{(x_n-x_1)(x_n-x_2)(x_n-x_3)\cdots(x_n-x_{n-1})}.$$

The first term on the right is equal to the sum of the partial fractions

$$\frac{1}{x_1-x_2}\left[\frac{1}{(x_2-x_3)(x_2-x_4)\cdots(x_2-x_n)}\right] + \frac{1}{x_1-x_3}\left[\frac{1}{(x_3-x_2)(x_3-x_4)\cdots(x_3-x_n)}\right] + \cdots + \frac{1}{x_1-x_n}\left[\frac{1}{(x_n-x_2)(x_n-x_3)\cdots(x_n-x_{n-1})}\right],$$

and hence the original sum is zero. This proves the first part. Since $\rho_j = 1/f''(\bar{x}_j)$, where $f'(\bar{x}_j)=0$, and the $\bar{x}_j$'s are distinct and real, the second part follows from the above on replacing $f(x)$ by $f'(x)$ with $n>2$.

Solved also by P. Chairulli, H. W. Eves, F. A. Ficken, P. M. Hummel, D. C. Lewis, R. K. Morley, and the proposer.

Editorial Note. Chiarulli's solution is similar to the above. The proposer's solution denotes by V the Vandermonde determinant of order n with the ith row $x_i^{n-1}, x_i^{n-2}, \cdots, x_i, 1$. In this determinant the cofactor of x_i^{n-1} is $(-1)^{i-1}V_i$, where V_i is a similar determinant obtained from V by omitting the first column and ith row. Then, taking unity as the leading coefficient of $f(x)$, $f'(x_i) = V/(-1)^{i-1}V_i$; and it follows that the sum of the reciprocals of $f'(x_i)$ is $1/V$ times the determinant obtained from V by replacing the elements of its first column by unity. Hence if $n \geqq 2$, this sum is zero. The solutions of Lewis and Morley consider the residues of $1/f(z)$, where $f(z)$ is a polynomial of the nth degree with distinct zeros z_i in the complex plane. The proofs are essentially as follows: The sum of the residues for the whole z-plane is zero, and the residue for the point at infinity is zero since $n \geqq 2$. It is obvious that the residue for the pole z_i is $1/f'(z_i)$, and this completes the proof of (a).

Hummel stated that the theorem of the problem results from the long known relations

$$(1)\qquad \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} = 0, \qquad k = 0, 1, \cdots, n-2; \qquad (2)\qquad \sum_{i=1}^{n} \frac{x_i^{n-1}}{f'(x_i)} = 1/c_0,$$

where $f(x)$ is a polynomial with complex coefficients, of which c_0 is the leading one, of degree $n \geqq 2$, and with distinct zeros x_i (see Burnside and Panton's *Theory of Equations*, vol. 1, p. 172, problem 4). The proof in this reference is different from his own which is briefly as follows: Assume that (1) is true for any chosen set of n distinct x_i's, let x_{n+1} be different from each of these, and set $F(x) = (x - x_{n+1})f(x)$. Then $F'(x_i) = (x_i - x_{n+1})f'(x_i)$, $i = 1, 2, \cdots, n$, and

$$0 = \sum_{i=1}^{n} \frac{x_i^k}{f'(x_i)} = \sum_{i=1}^{n} \frac{(x_i - x_{n+1})}{F'(x_i)}, \qquad k \leqq n-2.$$

We may then write

$$\sum_{i=1}^{n+1} \frac{x_i^{k+1}}{F'(x_i)} = x_{n+1} \sum_{i=1}^{n+1} \frac{x_i^k}{F'(x_i)},$$

and, since x_{n+1} may be any one of the $n+1$ distinct zeros of $F(x)$, we see at once that in this last equation the sum on the right and the one on the left must each be zero. Since (1) is easily verified for $n = 2$, the induction proof of it is now complete. Set $1 = nc_0x_i^{n-1}/f'(x_i) + (n-1)c_1x_i^{n-2}/f'(x_i) + \cdots$, sum each side for $i = 1, 2, \cdots, n$, and then (2) follows after using (1). The induction proof by Eves for the case $k = 0$ is essentially the same.

Ficken used Lagrange's interpolation formula on an auxiliary function and differentiation of the result gives

$$\sum_{i=1}^{n} \frac{\lambda_i}{x_i - x_0} + \frac{1}{f(x_0)} = 0, \qquad f(x_0) \neq 0, \qquad n \geqq 1.$$

Let $F(x) = (x - x_0)f(x)$, then $\Lambda_0 = 1/f(x_0)$, $\Lambda_i = \lambda_i/(x_i - x_0)$; and it follows that the sum of the Λ_i's is zero for $i = 0, 1, \cdots, n$, thus proving (a). The above formula results directly by the use of the partial fraction expansion of $1/f(x)$, then the proof is closely related to the one by Brock.

After the completion of his solution Lewis discovered that a generalization of the problem is given in Pólya und Szegö's *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 87, problem 67. He stated that his proof can be adapted to this generalization. The proof in this reference is essentially the same as the one in the first reference above.

A solution by Han Ming-tê, Peiping, China, was received after the preparation of the above. In the same cover with it were solutions of 3968 and 3969 by Li Ou using inversive geometry.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The members of the Mathematical Association are reminded that, as previously announced, the Annals of Mathematics and the Duke Journal of Mathematics are continuing the generous plan of allowing the half rate for those journals to those whose membership in the Association and whose subscription to the respective journals are unbroken from 1942 to the year in question.

Since the beginning of 1942, Professor G. D. Birkhoff has been acting as exchange professor with Latin-American universities. He has given series of lectures in the universities at Mexico City; Lima and Arequipa, Peru; and Santiago, Chile. At a special session he was made an honorary member of the faculty of the Universidad Nacional Mayor de San Marcos de Lima, which is the oldest institution of higher learning in the western hemisphere.

The University of Michigan announces the following promotions: Dr. C. C. Craig and Dr. R. V. Churchill have been promoted to professorships; Dr. P. S. Dwyer has been promoted to an associate professorship; Dr. R. M. Thrall has been promoted to an assistant professorship.

The University of Oregon has granted leaves of absence to Assistant Professors K. S. Ghent and T. S. Peterson for service with the Naval Ordnance Department.

Purdue University announces the following promotions and appointments: Dr. M. W. Keller and Dr. J. W. T. Youngs have been promoted to assistant professorships; Dr. Leonidas Alaoglu, B. H. Arnold, N. J. Fine, Dr. J. H. Giese, Dr. Michael Golomb, Dr. Ivan Niven, Dr. Maxwell Reade, and Dr. G. S. Young have been appointed as instructors.

At Washington and Jefferson College Associate Professor H. C. Shaub has been promoted to a professorship and made head of the department. D. T. Finkbeiner has been appointed to an instructorship.

Professor C. R. Adams has been made chairman of the department of mathematics in Brown University.

Professor C. B. Allendoerfer is on leave from Haverford College to become associate physicist in the Bureau of Ordnance, Navy Department.

Assistant Professor Frances E. Baker of Mount Holyoke College has been appointed to an associate professorship at Vassar College.

Assistant Professor P. N. Carpenter of Grove City College has been promoted to an associate professorship.

Cornell College, Iowa, has appointed W. M. Davis of the Illinois Institute of Technology to an assistant professorship.

Dr. D. M. Dribin of the University of Nebraska was appointed in June for work in cryptanalysis in the Intelligence Service, Office of the Chief Signal Officer in Washington.

Assistant Professor H. A. Giddings of the Illinois Institute of Technology has been appointed to an associate professorship at the University of New Hampshire.

Professor H. V. Gummere of Haverford College retired in June, at which time Haverford College conferred upon him the doctorate of science.

Dr. E. R. Hedrick retired on July 1 as vice-president of the University of California and provost at the Los Angeles campus.

Dr. P. R. Halmos has been appointed associate at the University of Illinois.

Dr. T. J. Higgins of Tulane University has been appointed an associate professor of electrical engineering at the Illinois Institute of Technology.

Professor M. H. Ingraham, head of the department of mathematics at the University of Wisconsin, has been appointed dean of the College of Letters and Science, and Professor R. E. Langer succeeds him as head of the department.

Professor S. C. Kleene of Amherst College is now a lieutenant (j.g.) in the U. S. Naval Reserve.

G. J. Neupert, head of the department of mathematics at Lewiston, Idaho, State Normal School has retired. He is succeeded by A. J. Boosinger who has been a member of the staff since last September.

Assistant Professor C. R. North of Rutgers University has been granted leave of absence to teach mathematics at the U. S. Naval Academy.

Dr. H. L. Olson has been appointed assistant professor at Southwestern University, Georgetown, Texas.

Professor H. A. Robinson of Agnes Scott College is teaching mathematics at the U. S. Military Academy, West Point, with the rank of major, F.A.

Assistant Professor E. A. Saibel of Carnegie Institute of Technology has been promoted to an associate professorship.

L. W. Swanson of the University of Minnesota has been appointed assistant professor at Coe College, Iowa.

At Fenn College, Dr. W. R. Van Voorhis has been promoted from instructor to assistant professor.

Dr. R. K. Wakerling of Texas Technological College has been appointed assistant professor at Fresno State College.

THE SIXTH CORPS AREA CONFERENCE ON PRE-INDUCTION TRAINING

On June 26 and 27, 1942, a conference representing colleges and universities in the Sixth Corps Area was held at Evanston, Illinois, to consider programs for pre-induction training of students who enlist for military service on a deferred basis. This conference was attended by over one hundred delegates from forty institutions, and by fifteen officers of the Army and the Navy. After a session devoted chiefly to a preliminary discussion by these officers of the problems before the conference, divisional meetings for mathematics and the sciences occupied the afternoon of the 26th, and most of the following morning. The various divisions then reported their recommendations to the conference as a whole.

The meetings of the mathematics division were attended by about fifty men representing more than twenty-five institutions. Lieutenant Commander E. P. Wilson was present and gave valuable advice and information.

At the afternoon session, Professor L. R. Ford presiding, discussion centered about the necessity or advisability of radical changes in the usual beginning college courses in mathematics. Accounts were given of programs that have been followed, or are planned for next year, at various institutions. A motion was passed expressing the sense of the meeting that it is feasible to meet the requirements of military pre-training programs, so far as Algebra and Plane Trigonometry are concerned, by suitable modification of existing courses, but it is desirable to organize a new course in Solid Geometry and Spherical Trigonometry. It was also recognized that something must be done to improve the technique of students in arithmetical computation.

Two committees were then appointed to draw up recommendations which might be presented to the conference as a whole. The first committee, on the program for a course in Solid Geometry and Spherical Trigonometry, consisted of Professors H. P. Evans (chairman), E. J. Moulton, and E. W. Schreiber. The second committee, whose members were Professors G. E. Moore (chairman), D. R. Curtiss, and E. B. Miller, prepared outlines of mathematical programs for each of several kinds of pre-induction training. These reports were presented and adopted by the division at the next morning's session, Professor C. N. Mills presiding, and were later adopted by the conference as a whole. The substance of these reports, as approved by the conference, follows.

The mathematics division recommends that courses be uniformly planned in the Sixth Army Corps Area as shown in the following chart. Since many institutions divide the academic year into semesters, while others use the quarter plan, this chart lists class hours in terms of both semester-hours (class hours per week for one semester) and quarter-hours (class hours per week for one quarter). Thus in the table below, Algebra (3s or 4q) indicates a course of 3 semester-hours, or one of 4 quarter-hours.

MINIMUM COURSE RECOMMENDATIONS

	Courses	Program Involved	
I	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (3) Solid Geometry and Spherical Trigonometry (3s or 4q)	Pre-Aviation Cadets as Bombardiers, Navigators, or Pilots	Flight
II	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (4) Analytic Geometry (3s or 4q) Calculus (8s or 12q in addition to above)	Communications Meteorology	Ground Forces
III	(1) Algebra (3s or 4q) (2) Plane Trigonometry (3s or 4q) (4) Analytic Geometry (3s or 4q) Calculus (6s or 8q in addition to above)	Geology Option Physics and Chemistry Option	Photography

Note 1. For course (1), students entering college with only one year of high school algebra should take a course in intermediate algebra; students entering with one and one half years should take a course in ordinary college algebra.

Note 2. In courses (1), (2), and (3), special emphasis should be placed on the use of tables, on interpolation, and on computation.

Note 3. Course (3), being of a combined character, was given special consideration by a separate committee. Details are found later in this report.

Note 4. Special emphasis should be placed on graphical methods in course (4), Analytic Geometry.

Note 5. Students who enter college with credit in College Algebra or Plane Trigonometry, or both, should be encouraged to take continuation work in mathematics beginning with their first semester in college. One suggestion is that, if they enter with two courses, they should take Analytic Geometry. If they enter with credit in College Algebra but not Trigonometry, under proper circumstances, they might continue with both Trigonometry and Analytic Geometry for groups II and III above.

Note 6. It is recommended to the supervising military authorities that the comprehensive examinations given in the various military programs cover in algebra just those topics which are common to the two courses: Intermediate Algebra and College Algebra.

COMPUTATIONAL ARITHMETIC

In view of the reiterated plea from representatives of the armed forces that we strive for greater accuracy and speed in computational arithmetic, we recommend that colleges give special drill work to improve computation. This may be done by definite drill in connection with courses in Algebra, Trigonometry, and Solid Geometry—or, schools may wish to offer special course work designed to remedy the serious lack of computational ability.

In the report of the committee, topical outlines of the courses were presented. Since these followed traditional subject outlines of such courses, only the following for the new course on Solid Geometry and Spherical Trigonometry is here given.

SOLID GEOMETRY AND SPHERICAL TRIGONOMETRY

Credit, 3 semester-hours or 4 quarter-hours. About 43 class periods.

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In the presentation of these topics emphasis should be placed on the making and interpretation of drawings of three-dimensional figures and the development of space intuition. Statements of theorems should be clarified by construction and use of models. When possible, the drawings of figures should be motivated by word problems involving mensuration formulas (the more important of these formulas should be memorized). The problems assigned should include many designed to give practice in computation; some should require the use of trigonometric functions and of logarithms. Problems concerning loci, frustums, and prismatoids might also be included.

The Sphere and Trihedral Angles. 10 lessons.

Definitions. Circles on a sphere. Area and volume of a sphere. Spherical angles. Spherical triangles and trihedral angles. Area of a spherical triangle.

Spherical Trigonometry and Applications.

Spherical Trigonometry. 10 lessons.

Polar triangles. Formulas for right spherical triangles. Napier's rules. Rules regarding the relative magnitude of parts. Solution of right spherical triangles, isosceles triangles, quadrantal triangles. The law of sines. The law of cosines. Formulas for half-angles and half-sides. Napier's analogies. Solution of oblique spherical triangles, six cases.

Applications of Plane and Spherical Trigonometry to Navigation. 8 lessons.

Plane sailing. Parallel sailing. Middle latitude sailing. Dead reckoning. Great circle sailing. Positions on the celestial sphere. The astronomical triangle, with applications.

A limited number of reports of the conference is available on application to the President of Northwestern University. Mimeographed accounts of the complete agenda of the mathematics division can be had, not more than one to an institution, and so long as they last, by writing to the Department of Mathematics, Northwestern University, Evanston, Illinois.

D. R. CURTISS

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.
Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 23–24, 1942
ILLINOIS
IOWA
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Dec. 5, 1942
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI, fall, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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